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Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields

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The spontaneous generation of reversed fields in toroidal plasmas is shown to be a consequence of relaxation under constraints. With perfect conductivity a topological constraint exists for each field line and the final state is not unique. With small departures from perfect conductivity, topological constraints are relaxed and the final state becomes unique. The onset of the reversed field and other features of this model agree well with observations on ZETA.

One of the best known methods for magnetic confinement of plasma is the toroidal pinch as in ZETA.[1,2] In such experiments a toroidal field B_0 is created by external coils and a toroidal current I is induced in the plasma. The pinch effect associated with this current produces the plasma compression whose magnitude depends on the ratio $2I/aB_0 \equiv \theta$ (where a is the minor radius of the torus).

A remarkable feature of these experiments is that after an initial, violently unstable, phase the plasma frequently relaxes into a "quiescent" state in which it appears to be largely stable. Furthermore, when the pinch ratio θ exceeds some critical value this relaxation is accompanied by the generation of a reversed toroidal field in the outer regions of the plasma.

This Letter outlines a theory of the relaxation of toroidal plasma which appears to account well for this remarkable behavior and which predicts the critical value of θ for the generation of the reversed field. It can also account for other phenomena observed in toroidal pinches.

In this theory the plasma is regarded as a conducting but viscous fluid enclosed in a rigid, perfectly conducting, toroidal vessel. The initial state is arbitrary except that both the magnetic field and current are tangential to the conducting wall. The system is *not* in stable equilibrium and when released will therefore move (usually violently) and dissipate energy before coming to rest. Only when its energy is a minimum is it incapable of further rapid movement. Hence the final state must be one which makes the energy a minimum subject to any constraints which are imposed on the allowed motion.[3] The major problem, of course, lies in determining and applying these constraints.

For simplicity it is assumed here, as is indeed the case in most experiments, that the plasma internal energy is negligible compared to the magnetic energy $W_m = \int (B^2/2) d\tau$ which is therefore to be minimized. The inclusion of plasma energy will be discussed elsewhere

The constraints which must be applied to the variations in $\vec{B}$ (without which the minimum would be $\boldsymbol{B}=0$) arise from the fact that in a perfectly conducting fluid variations in the magnetic field must satisfy

$$\partial \vec{B}/\partial t - \nabla \times (\vec{v} \times \vec{B}) = 0, \qquad (1)$$

where $\vec{v}$ is the fluid velocity.

As is well known, Eq. (1) means that lines of force may be labeled by the fluid elements on them and so be regarded as moving with the fluid velocity. Since this velocity is continuous, field lines cannot break or coalesce (except where $B=0$ which we exclude). All topological properties of the field lines are therefore invariant; e.g., if two closed field lines are initially linked n times, then they must remain so linked at all times.

These topological constraints can be expressed through the vector potential, $\vec{B}=\nabla\times\vec{A}$. From Eq. (1) this must satisfy

$$\partial\vec{A}/\partial t=\vec{v}\times\vec{B}+\nabla\chi, \tag{2}$$

where χ is an arbitrary gauge. Since the energy is to be minimized over all fluid motions it is clear that Eq. (2) imposes no restriction on the component of $\partial\vec{A}/\partial t$ perpendicular to $\vec{B}$. It might appear that the parallel component is also unrestricted on account of the arbitrary gauge. However χ must satisfy the magnetic differential equation

$$\vec{B}\cdot\nabla\chi=\vec{B}\cdot\partial\vec{A}/\partial t,$$

and is therefore single-valued only if $\partial\vec{A}/\partial t$ is constrained so that

$$\oint\frac{dl}{B}\vec{B}\cdot\frac{\partial\vec{A}}{\partial t}\ \text{and}\ \oint\frac{dS}{|\nabla\Psi|}\vec{B}\cdot\frac{\partial\vec{A}}{\partial t},$$

vanish on any closed field line and on any magnetic surface, respectively.

These constraints can be summarized as follows: For every volume V bounded by field lines the quantity

$$K\equiv\int_V\vec{A}\cdot\vec{B}\,d\tau, \tag{3}$$

is an invariant of the motion. When the field lines are closed there is one invariant for each line (the volume V then being an infinitesimal flux tube); if the field generates magnetic surfaces there is one invariant for each surface. The invariance of K for any volume V bounded by a flux surface follows directly from Eqs. (1) and (2). Furthermore the argument can be reversed to show that if K is invariant for every flux tube then $\partial\vec{A}/\partial t$ is expressible in the form $\vec{u}\times\vec{B}+\nabla\varphi$. Consequently all the restrictions on $\partial\vec{A}/\partial t$ imposed on Eq. (1) are embodied in the invariants. [Note however that $\vec{u}$ need not be the same as the fluid velocity $\vec{v}$; this is because the velocity of lines of force is not unique.[4]]

The state in which the magnetic energy is a minimum for all variations $\delta\vec{A}$ which leave the invariants unchanged is given by

$$\nabla\times\vec{B}=\lambda(a,b)\vec{B}, \tag{4}$$

where $\lambda(a,b)$ is any function constant along field lines ($\vec{B}\cdot\nabla\lambda=0$). Hence when all the constraints appropriate to a perfectly conducting fluid are observed, the state of minimum magnetic energy is some force-free configuration. Exactly which force-free configuration can only be found by determining λ from the initial values of the invariants K.

Now let us consider how the situation is modified by small departures from the perfect-conductivity approximation of Eq. (1). It is well known, from studies of resistive instabilities[5,6] and elsewhere, that the main consequence of any small departure from perfect conductivity is that topological properties of the magnetic field are no longer preserved and lines of force may break and coalesce. It seems inevitable that during the violent phase of the diffuse pinch, resistivity, inertia, microturbulence, or some other departure from perfect conductivity will bring about such a relaxation of the topological constraints.

Once lines of force may break and coalesce it clearly makes no sense to suggest that $\int\vec{A}\cdot\vec{B}$ be an invariant for each line of force! However, changes in field topology are accompanied by only very small changes in the field itself and the sum of $\int\vec{A}\cdot\vec{B}$ over all field lines will be almost unchanged so long as departures from perfect conductivity are small. The effect of the topological changes is merely to redistribute the integrand among the field lines involved. Thus the integral $\int\vec{A}\cdot\vec{B}$ over the *total* volume of the system will still be a good invariant even though $\int\vec{A}\cdot\vec{B}$ on each flux tube certainly is not.

The final state of relaxation, therefore, will now be the state of minimum energy subject only to the single invariant

$$K_0=\int_{V_0}\vec{A}\cdot\vec{B}\,d\tau, \tag{5}$$

where V_0 is the total volume of the system. This state is easily determined, and is given by[7]

$$\nabla\times\vec{B}=\mu\vec{B}, \tag{6}$$

where μ is now a single constant having the *same* value on all field lines. Thus, when topological constraints are relaxed, the final state is no longer *any* force-free configuration but a specific one completely determined once μ and the scale factor multiplying the solution of Eq. (6) are known. These are obtained as follows. The solution of Eq. (6) is $\vec{B}=B_0\vec{h}(\mu r,\mu a,\mu R)$, where $\vec{h}$ is the normalized solution and a,R are the dimensions of the toroidal container. Hence the invariant takes the form $K_0=(B_0^2/\mu^4)g(\mu a,\mu R)$ and the toroidal flux Ψ, which is also invariant, takes the form $(B_0/\mu^2)f(\mu a,\mu R)$. The value of μ is therefore obtained from the ratio K_0/Ψ^2. Clearly K_0 and Ψ together completely determine the final state. [It may be shown that K_0/Ψ is essentially the "volt-seconds" stored in the plasma so that μ is related to the ratio (stored volt-seconds)/(toroidal flux).]

One may now return to the question of the appearance of the reversed toroidal magnetic field in the relaxed state. As has been shown, for a given toroidal aspect ratio this can depend only on the value of μ. If the toroidal curvature is neglected the axisymmetric solution of Eq. (6) is $B_z = B_0 J_0(\mu r)$, $B_\theta = B_0 J_1(\mu r)$ (where r, θ, z are cylinder coordinates). Reversal of the toroidal (B_z) field therefore occurs when $\mu a > 2.404$. In terms of the experimental pinch parameter θ this point corresponds to $\theta = 1.202$, which is in satisfactory agreement with the value $\theta \simeq 1.4$ observed for the onset of the reversed field in ZETA.[8]

It should be noted that although the state of minimum energy is described by Eq. (6), it does not follow that the axisymmetric solution must be this state; for Eq. (6) may have other solutions with lower energy.[9] This point will be discussed in detail elsewhere. It turns out that the axisymmetric solution *is* indeed the lowest energy state compatible with specified values of K_0 and Ψ, when K_0/Ψ^2 is less than a critical value, but beyond this value the lowest energy state is a helical solution. The transition occurs at $\mu a = 3.11$ (or $\theta \simeq 1.6$) which is also the boundary for resistive instability of the axisymmetric state.[10] However, unlike linear instability theory, the present theory determines the amplitude of the helical deformation and the change in θ which accompanies the transition from axisymmetric to helical state.

In conclusion, the behavior of toroidal pinches is well accounted for by the general principle that plasmas relax to a state of minimum energy subject to all relevant constraints. If the plasma were *perfectly* conducting, lines of force would preserve their identities and there would be a topological constraint associated with each field line. In this case the final state may be any equilibrium. However in the presence of small departures from perfect conductivity, topological constraints on lines of force are relaxed; they no longer retain their identity and consequently only one invariant remains. The final state is then a unique configuration depending only on the ratio K_0/Ψ^2 or equivalently on the pinch ratio θ. When θ exceeds ~1.2 this final state is one with a reversed toroidal field, in agreement with observation. When θ exceeds a second critical value ~1.6 the final state is helically deformed.

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