[6]G. Kalman, P. Bakshi, and R. Cover, in *Cosmic Plasma Physics*, edited by K. Schindler (Plenum, New York, 1972), p. 261; R. Cover and G. Kalman, to be published; R. Cover, G. Kalman, and P. Bakshi, Bull. Amer. Phys. Soc. 18, 1266 (1973).

[7]P. A. Sturrock, Astrophys. J. 164, 529 (1971).

[8]J. Geheniau, Physica (Utrecht) 16, 822 (1950); J. Geheniau and M. Demeur, Physica (Utrecht) 17, 71 (1951).

[9]F. Pacini, in Proceedings of the Sixteenth Solvay Conference, 1973 (to be published).

[10]R. Baier and P. Breitenlohner, Nuovo Cimento 47, 261 (1967).

[11]H. Euler, Ann. Phys. (Leipzig) 26, 398 (1936); H. Euler and B. Kockel, Naturwissenschaften 23, 246 (1936); W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936); V. S. Weisskopf, Kgl. Dan. Vidensk. Selsk., Mat.-Fys. Medd. 14, No. 6 (1936).

[12]W.-y. Tsai and A. Yildiz, Phys. Rev. D 8, 3446 (1973).

[13]B. Jancovici, Phys. Rev. 187, 2275 (1969); R. G. Newton, Phys. Rev. D 3, 626 (1971). See also H. Y. Chiu, V. Canuto, and L. Fassio-Canuto, Phys. Rev. 176, 1438 (1968), for the initial attempt.

Conversion of Electromagnetic to Gravitational Radiation by Scattering from a Charged Black Hole

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(Received 16 September 1974)

When a purely electromagnetic incoming wave is scattered by a charged black hole, the electromagnetic-gravitational coupling can lead to conversion to gravitational radiation in the outgoing wave. Analytic estimates and numerical results are given for this effect, which proves to be significant only for maximally or near-maximally charged black holes.

Gerlach[1] has pointed out that in the presence of a background electromagnetic field an electromagnetic wave will gradually convert into a gravitational wave and vice versa. In particular he notes that an electromagnetic wave scattering from a charged black hole could convert entirely into gravitational radiation. Zerilli[2] and Moncrief[3,4] have found the coupled and decoupled wave equations obeyed by electromagnetic and gravitational perturbations on a Reissner-Nordstrøm charged-black-hole-background space-time for both odd- and even-parity waves.

A purely electromagnetic incoming wave is a linear combination of two normal modes. Because the normal modes obey different wave equations, their relative phase will be shifted by scattering from the black hole. The outgoing wave will be a combination of both electromagnetic and gravitational radiation. In this note we give the results of computer calculations of this effect for the odd-parity perturbations.

After separation of harmonic time and angular dependence, the gauge-invariant Moncrief radial equations for odd-parity normal modes are

$$(d^2/dr^{*2}+\omega^2-V_\pm)R_\pm=0, \qquad (1)$$

where

$$V_\pm=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)\left(\frac{L(L+1)}{r^2}-\frac{3M}{r^3}+\frac{4Q^2}{r^4}\pm\frac{\sigma}{r^3}\right),$$

and where $\sigma=[9M^2+4Q^2(L-1)(L+2)]^{1/2}$, M and Q are the mass and charge of the black hole,[5] ω is the angular frequency of the wave, and L is the total angular momentum index in the Regge-Wheeler tensor harmonic expansion. The r^* coordinate is defined by $dr/dr^*=1-2M/r+Q^2/r^2$. V_+ and V_- are the effective potentials, which vanish at infinity and on the horizon at $r=M+(M^2-Q^2)^{1/2}$ (or $r^*=-\infty$). The barrier maximum for each effective potential occurs near $r=[3M+(9M^2-8Q^2)^{1/2}]/2$.

The normal modes R_+ and R_- are defined by

$$R_+=F\cos\Psi+G\sin\Psi,$$
$$R_-=-F\sin\Psi+G\cos\Psi,$$
$$\sin2\Psi=2Q[(L-1)(L+2)]^{1/2}/\sigma,$$

in terms of Moncrief's gauge-invariant amplitudes. F represents a purely electromagnetic perturbation, and G represents a combination which near infinity is purely gravitational.[6] With time dependence $e^{-i\omega t}$, and with the boundary condition of purely ingoing waves at the horizon, the normal modes take the asymptotic form

$$R_\pm \to \begin{cases} \exp(-i\omega r^*) + A_\pm \exp(i\omega r^*), & r^* \to +\infty, \\ B_\pm \exp(-i\omega r^*), & r^* \to -\infty. \end{cases}$$

For a wave whose incoming part is purely electromagnetic, the amplitude of the outgoing gravitational radiation at infinity is given by

$$G_{\rm out} = F_{\rm in} \sin(2\Psi)(A_+ - A_-)/2.$$

If

$$A_\pm = \lambda_\pm \exp[-i(\Omega \pm P)],$$

the fractional conversion of electromagnetic to gravitational radiation by the black hole is given by

$$C = |G_{\rm out}|^2/|F_{\rm in}|^2$$
$$= \sin^2(2\Psi)[\tfrac{1}{4}(\lambda_+ - \lambda_-)^2 + \lambda_+\lambda_- \sin^2 P]. \tag{2}$$

Evidently, in order to attain $C \approx 1$ (full conversion to outgoing gravitational waves) three conditions must be met: $QL \gg M$, so that $\sin^2(2\Psi) \approx 1$; $\lambda_+ \approx 1$ and $\lambda_- \approx 1$, so that both modes will be reflected back to infinity; and finally, $P \approx (2n+1)\pi/2$.

The barrier-reflection factors $\lambda_\pm$ and the relative phase shift P are determined by using the WKB approximation (valid for $L \gtrsim 10$) to solve Eqs. (1). The barrier penetration is evaluated by using parabolic cylinder functions[7] with the parabolic potential fitted to the top of the barrier. The equations thus obtained for P and $\lambda_\pm$ are

$$P = \int_{r^*(r_-^t)}^{\infty} (\omega^2 - V_-)^{1/2} dr^* - \int_{r^*(r_+^t)}^{\infty} (\omega^2 - V_+)^{1/2} dr^* + \tfrac{1}{2}\mathrm{Im}\ln[\Gamma(\tfrac{1}{2} + ia_-)/\Gamma(\tfrac{1}{2} + ia_+)]$$
$$+ \tfrac{1}{2}(a_+ \ln a_+ - a_- \ln a_-) - \tfrac{1}{2}(a_+ - a_-), \tag{3}$$

$$\lambda_\pm = [1 + \exp(-2\pi a_\pm)]^{-1/2}, \tag{4}$$

with

$$a_\pm = \left.\frac{V_\pm - \omega^2}{(-2\,d^2V_\pm/dr^{*2})^{1/2}}\right|_{r = r_\pm^m}.$$

At $r = r_\pm^m$ the function $V_\pm$ attains its maximum value $V_\pm^m$, while $r_\pm^t$ is the value of r at which $\omega^2 = V_\pm$.

The numerical calculations use Eqs. (3) and (4). However, certain analytic approximations can be made to show more clearly the dependence of the phase shift on the parameters Q, M, L, and ω.

As a function of ω/L the phase integral, Eq. (3), becomes to lowest order in $1/L$

$$P \approx \frac{\sigma}{L} \int_{r^t}^{\infty} \frac{dr}{r^2[r^2(\omega/L)^2 - (1 - 2M/r + Q^2/r^2)]^{1/2}}, \tag{5}$$

with r^t being the value of r for which the integrand diverges. This approximation is valid for $L \gg 1$ and ω^2 not too near the top of the barrier. More precisely, ω^2 must be such that $(\omega^2 - V_-^m)/(\omega^2 - V_+^m) \approx 1$, or equivalently, $1 - (\omega/\omega_0)^2 \gtrsim \sigma/ML^2$, where ω_0 is defined by $\omega_0^2 = (V_-^m + V_+^m)/2$.

In the frequency regime where $\omega M/L \ll 1$, Eq.

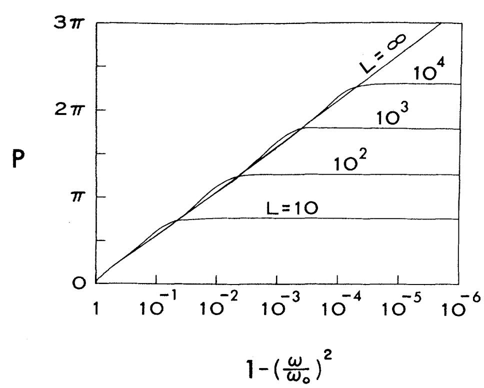

FIG. 1. Relative phase shift for a black hole with $Q/M = 1$, shown as a function of a frequency variable which expands the region near the potential maxima. The square of the frequency, $\omega_0^2 = (V_-^m + V_+^m)/2$, lies half-way between the two potential maxima.

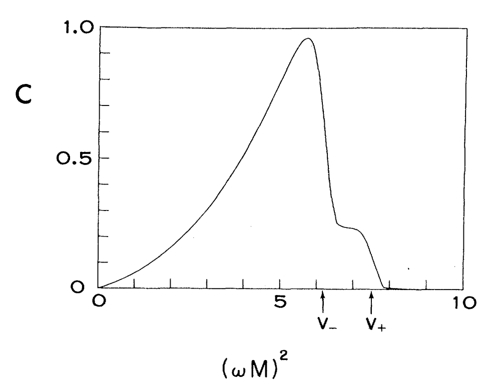

FIG. 2. Conversion factor for $Q/M = 1$, $L = 10$, shown as a function of the square of the frequency of the wave (in units of M^{-2}). $C = 1$ represents full conversion to gravitational radiation; $C = 0$, no outgoing gravitational radiation. Arrows indicate the frequencies corresponding to the potential maxima.

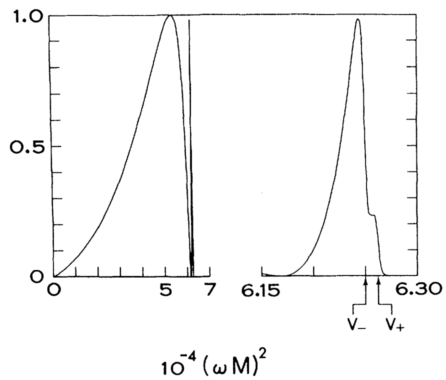

FIG. 3. Conversion factor for $Q/M=1$, $L=10^3$, shown as a function of the square of the frequency of the wave (in units of $10^4 M^{-2}$). The region near the potential maxima is repeated with an expanded frequency scale.

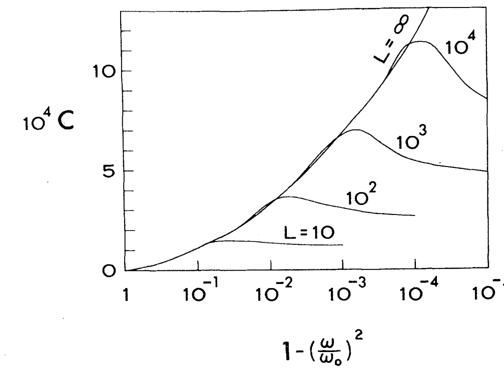

FIG. 4. Conversion factor (in units of 10^{-4}) for a black hole with $Q/M=0.01$, shown as a function of a frequency variable which expands the region near the potential maxima.

(5) becomes

$$P \approx \sigma\omega/L^2. \tag{6}$$

As this is much less than unity, and as both λ_+ and λ_- are approximately unity in this low-frequency limit, the conversion factor becomes

$$C \approx 4Q^2\omega^2/L^2. \tag{7}$$

In the frequency regime where $1 \gg 1-(\omega/\omega_0)^2 \gtrsim \sigma/ML^2$, the dominant contribution to the integral in Eq. (5) comes from the region near the turning point r^t. A parabolic approximation to the potential in this region gives

$$P \approx -\frac{\sigma \ln[1-(\omega/\omega_0)^2]}{L[9M^2-8Q^2]^{1/4}[6M+2(9M^2-8Q^2)^{1/2}]^{1/2}}. \tag{8}$$

Figure 1 gives the results of a computer evaluation of Eq. (3) for $Q/M=1$ and for various values of L. Significant conversion of electromagnetic to gravitational radiation occurs for $Q/M=1$, even with moderate values of L (Fig. 2). As L increases there are additional cycles of conversion [$C=1$, $P=(2n+1)\pi/2$] and reconversion ($C=0$, $P=n\pi$), as shown in Fig. 3. The analytic approximations, Eqs. (6)–(8), are seen to be accurate in their regions of validity.

For ω^2 near to or over the top of the potential barriers, the transmission of the modes into the black hole dominates the behavior of the conversion factor, as a function of frequency. As ω^2 passes the value V_-^m, λ_- falls rapidly to zero, implying complete transmission of the minus mode into the black hole. For $Q/M=1$, the plus mode is still reflected back to infinity ($\lambda_+ \approx 1$), leading to the $C \approx \frac{1}{4}$ shoulder (Figs. 2 and 3) for frequencies between the potential maxima.

For $Q \ll M$, the conversion factor becomes very small for almost all values of ω and L. Using Eq. (8) one finds that only for $L \gtrsim (Q/M)e^{3\pi M/2Q}$ can the phase shift approach $\pi/2$ and the conversion factor approach unity. Furthermore these large phase shifts occur for very large frequencies ($\omega \sim L/M$) and over very narrow frequency ranges ($\Delta\omega/\omega \sim Q/LM$).

For the case $Q/M=0.01$, computer calculations given in Fig. 4 show that the conversion factor is less than 10^{-3} for all values of ω with $L<10^4$, consistent with estimates based on Eqs. (2) and (8). Only for $L \gtrsim 10^{200}$ would one obtain conversion factors of order unity.

The very narrow frequency range and the high frequencies necessary rule out this process of conversion of electromagnetic to gravitational radiation (or vice versa) for any but $Q \sim M$ black holes. If solar-mass or larger black holes have small values of Q/M (for example, because of neutralization of the charge by infall of surrounding matter), then this process will have insignificant astrophysical consequences.

The authors thank V. Moncrief for many conversations and much helpful advice.

*Work supported in part by the National Science Foundation under Grant No. GP-29257.

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[1]U. H. Gerlach, Phys. Rev. Lett. 32, 1023 (1974).
[2]F. Zerilli, Phys. Rev. D 9, 860 (1974).
[3]V. Moncrief, Phys. Rev. D 9, 2707 (1974).
[4]V. Moncrief, Phys. Rev. D 10, 1057 (1974).
[5]Units are chosen such that $G=c=1$.
[6]Our amplitudes F and G are the same as Moncrief's $\hat{\pi}_f$ and $\hat{\pi}_g$, respectively.
[7]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), Chap. 19.

Production of Neutral Weak Bosons in High-Energy Electron and Muon Experiments*

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We discuss the theoretical total and differential cross sections for Z^0 production in the reaction $l+N\rightarrow N'+Z^0+l$, including remarks about polarization effects and subsequent leptonic decays. Brief mention is made about production of Z^0's by neutrino beams and of scalar mesons.

As experimental evidence for neutral weak currents continues to mount,[1] the neutral intermediate weak boson Z^0 appears to have gained equal footing with the other hypothetical spin-1 bosons $W^{\pm}$. Of course, the possibility that all three exist is strong in view of the prominent place which gauge theories hold in our present thinking about weak forces. It is fair to say that searches for Z^0 are of paramount importance in attempts to corroborate these ideas about the hierarchy of elementary interactions.

We describe here one attractive candidate reaction for this search, namely

$$l^{\pm}+N\rightarrow l^{\pm}+N'+Z^0. \tag{1}$$

Specific attention is paid to the diagrams of Figs. 1(a) and 1(b) for weak "bremsstrahlung" by the lepton beam during electromagnetic scattering off of some nucleus N. The advantages of this particular mode lie in the fact that we can avoid certain large backgrounds present in searches via proton-proton collisions[2] and in the fact that we now have very high-energy lepton beams available. There is also no need to home in on a narrow resonance peak in contrast to production via $e^{\pm}$ colliding beams.

High-energy muons at Fermi National Accelerator Laboratory are the lepton beams we have uppermost in mind. It is interesting that the decided energy advantage of muons over neutrinos (from pion decay) is *not* ruined here for any dynamical reason. That is, we saw a few years ago[3] that theoretical cross sections for $W^{\pm}$ production by neutrino beams were 2 orders of magnitude larger on the average than those involving muon beams. The key to this difference is whether or not there is a muon in the final state; after a cancelation in a gauge-invariant set of graphs, the final muon propagator dominates. Thus, Reaction (1) gets the same enhancement seen in $\nu+N\rightarrow\mu+N'+W$, a circumstance which motivated the work described in this note.

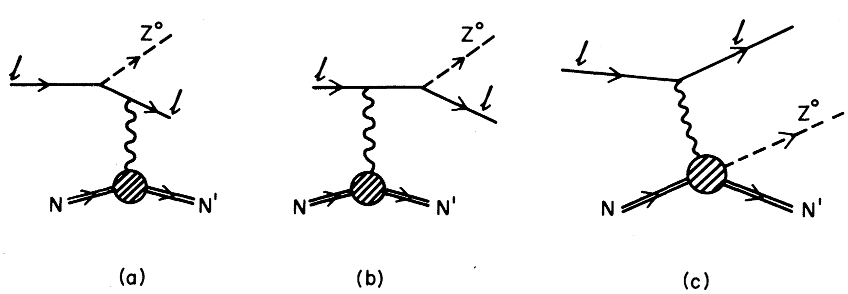

FIG. 1. Feyman diagrams for the production of Z^0 by leptons in the electromagnetic field of some nucleus.