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Type-N Gravitational Field with Twist

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I have found a type-N gravitational field for which the principal null direction has nonzero twist and which admits exactly one Killing vector.

In a recent review[1] of the known exact solutions of the Einstein field equations, Kinnersley pointed out that not a single type-N solution with twisting rays[2] was known, though the type-N problem for zero twist had been completely solved[3] over a decade ago. I have now obtained a particular type-N solution with twist. It is my hope that this new solution will further the understanding of gravitational waves with twisting rays. I will now describe the derivation, except for some lengthy techniques which can be extended to the search for other type-N solutions with twist and which will be published elsewhere.

The derivation employs a null tetrad[4] which consists of one-forms k, m, t, t^* such that k is a principal null vector, k and m are real, t^* is the complex conjugate of t, and $k\cdot m = t\cdot t^* = 1$. For a type-$N$ vacuum, this null tetrad can be chosen so that the corresponding connection forms have components

$$(\nabla_\alpha k_\beta)t^\beta = \nabla_\alpha \zeta = z(t_\alpha + Ak_\alpha), \tag{1}$$

$$(\nabla_\alpha k_\beta)m^\beta = (\nabla_\alpha t_\beta)t^{\beta *} = 0, \tag{2}$$

$$(\nabla_\alpha m_\beta)t^{\beta *} = h\nabla_\alpha \zeta, \tag{3}$$

where ζ, z, A, and h are complex scalar fields. The real and imaginary parts of z are the divergence and the twist, respectively. Let

$$z^{-1} = u + i\tau, \quad \sqrt{2}\,\zeta = \rho + i\,\sigma, \tag{4}$$

where u, τ, ρ, and σ are real. Then Eqs. (1) and (2) imply that the two-form dk is given by

$$dk = k(A^* d\zeta + A\,d\zeta^*) - 2\tau\, d\rho\, d\sigma. \tag{5}$$

The particular case for which I have found a solution is defined by the statement that the first term on the right-hand side of Eq. (5) is equal to

$$k(A^* d\zeta + A\,d\zeta^*) = k\,d\chi, \tag{6}$$

where

$$d\chi = -d\tau/\tau. \tag{7}$$

Let

$$\Delta = e^\chi \tau. \tag{8}$$

Then Eq. (7) is equivalent to the statement that Δ is a uniform field, and Eqs. (5) and (6) imply $d(e^\chi k + 2\Delta\rho\, d\sigma) = 0$. Hence, there exists a scalar field ξ such that $d\xi = e^\chi k + 2\Delta\rho\, d\sigma$.

ρ, σ, ξ, and u serve as our coordinates. The solution is given by

$$k = e^{-\chi}(d\xi - 2\Delta\rho\, d\sigma), \tag{9}$$

$$m = du + 3i\tau(A^* d\zeta - A\,d\zeta^*), \tag{10}$$

$$t = z^{-1} d\zeta - Ak, \tag{11}$$

$$\tau = \Delta e^{-\chi} = \Delta\rho^{3/2} f(y), \quad y = \xi/\Delta\rho^2, \tag{12}$$

$$A = \sqrt{2}\rho^{-1}[(y-i)f^{-1}f' - \tfrac{3}{4}], \tag{13}$$

$$h = (3i\Delta/4)(\xi - i\Delta\rho^2)^{-1}, \tag{14}$$

where f is a function of y, $f' = df/dy$, such that

$$f'' + [\tfrac{3}{16}(1+y^2)]f = 0. \tag{15}$$

The only nonzero components of the Riemann tensor are given by

$$m^\alpha t^{\beta *} m^\gamma t^{\delta *} R_{\alpha\beta\gamma\delta} = 4ze^\chi\, \partial h/\partial\xi. \tag{16}$$

Since none of the scalar coefficients in Eqs. (9) to (11) depends on σ, the tangent vector $\partial/\partial\sigma$ is a Killing vector.

Any even solution of Eq. (15) has zeros only at $y = \pm 5.5$, as determined by a numerical integration; any odd solution has only the one zero at $y = 0$. The asymptotic form of any solution of Eq. (15) at large y is a constant times $y^{3/4}$ or $y^{1/4}$.

If the function f is such that arbitrarily large $|y|$ are admissible in our coordinate system, then we fix the multiplicative factor in f (and, therefore, in Δ, ξ, and $\exp\chi$) by specifying that the asymptotic values be $|\Delta y|^{3/4}$ or $|\Delta y|^{1/4}$. Then, if the asymptotic form is $|\Delta y|^{3/4}$, the limiting case $\Delta = 0$ corresponds to a Minkowski space with Cartesian coordiantes x_α which are related to ours by $2u = x_3 - x_4$, $u\rho = x_1$, $u\sigma = x_2$, and $2v = 2(x_3 + x_4) + u(\rho^2 + \sigma^2)$, where $v = 4\xi^{7/4}/7$. Similar relations hold when the asymptotic form is $|\Delta y|^{1/4}$.

I thank Dr. Fred Ernst for verifying the solution, for the numerical integration of Eq. (15), and for the relations of the coordinates to Cartesian coordinates when $\Delta = 0$.

[1]W. Kinnersley, in Proceedings of the Seventh International Conference on Gravitation and General Relativity, Tel Aviv, Israel, 24–28 June 1974 (to be published).

[2]For a review and bibliography on algebraically special gravitational fields with twisting rays, see Ref. 1. Equation (49) of Ref. 1 contains relatively simple forms of the type-N field equations due to A. Exton (private communication).

[3]See for example, I. Robinson and A. Trautman, Phys. Rev. Lett. 4, 431 (1960); W. Kundt, Z. Phys. 163, 77 (1961); I. Robinson and A. Trautman, Proc. Roy. Soc., Ser. A 265, 463 (1962).

[4]Our notation for the null tetrad is used, e.g., by R. K. Sachs, in *Relativity, Groups, and Topology*, edited by C. DeWitt and B. deWitt (Gordon and Breach, New York, 1964), p. 530. Our signature is +2.

Longitudinal, Massive Photon in an External Magnetic Field

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The study of the vacuum polarization for strong magnetic fields reveals the existence of a massive, longitudinal photonlike resonance for magnetic fields exceeding $B \simeq m^2c^4/e^3$.

The existence of strong magnetic fields in the vicinity of collapsed stars has brought into focus the question of the dispersive properties of the vacuum in the presence of strong magnetic fields.[1-7] Such dispersive features become significant when the magnetic field reaches and exceeds the critical field value $B_* = m^2c^3/\hbar e = 4.41 \times 10^{13}$ G. In this Letter a novel feature of the electromagnetic vacuum is pointed out: When the magnetic field strength exceeds a second critical value B_0 of the order of $(\hbar c/e^2)B_*$, a massive, longitudinal photonlike resonance appears whose mass value roughly coincides with the energy of an electron-positron pair occupying the lowest Landau level. For magnetic field values in the vicinity of the critical field the photon is heavily damped, but for increasing field strengths the lifetime rapidly increases: Typically for $\boldsymbol{B} \simeq 10B_0$, $\tau \simeq 2\pi \times 15 \times \omega^{-1} \simeq 7 \times 10^{-18}$ sec.

The existence of the massive photon is inferred from the study of the longitudinal dispersion relation

$$\epsilon(\vec{k}, \omega) = 0, \tag{1}$$

with $\epsilon(\vec{k}, \omega)$, the longitudinal "dielectric function," and $\alpha(\vec{k}, \omega)$, the longitudinal polarizability of the vacuum, being related to the regularized longitudinal vacuum polarization $\overline{\Pi}_{33}(\vec{k}, \omega)$ by

$$\epsilon(\vec{k}, \omega) \equiv 1 + \alpha(\vec{k}, \omega) = 1 - \omega^{-2}\overline{\Pi}_{33}(\vec{k}, \omega). \tag{2}$$

We calculated the vacuum polarization in the presence of an arbitrarily strong, uniform magnetic field, to lowest order in e^2. The photon momentum is restricted to be parallel to $\vec{B}$. For small momenta the k dependence of Π_{33} is not significant and in this Letter we consider the $k \to 0$ limit only. Π_{33} can be obtained by standard methods, using the appropriate Green's functions for electrons in a magnetic field.[8] The unrenormalized value of Π_{33} is given by

$$\Pi_{33}(0, \omega) = 8\frac{e^3B}{2\pi} \sum_{n=0}^{\infty} a_n \int_{-\infty}^{\infty} \frac{m^2 + 2neB}{\epsilon_{pn}(\omega^2 - 4\epsilon_{pn}^2)}\, dp, \tag{3}$$