Parametric Instabilities in Finite Inhomogeneous Media

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The general first-order parametric-instability equations for an inhomogeneous finite-size plasma are solved both analytically and numerically. It is shown that the wave-number mismatch introduced by plasma gradients does *not* necessarily convert absolute instability into convective instability, even for large systems. Absolute instability can also occur when the length of the system is determined by a smooth spatial dependence of the growth rate.

Recently, there has been considerable interest in the properties of parametric instabilities in inhomogeneous media[1-4] because of their importance in the anomalous absorption[5] or scattering[6] of laser light in the low-density corona of a laser-irradiated DT pellet.[7] Using an unbounded but inhomogeneous plasma model, Rosenbluth[2] and Piliya[3] have shown that in the presence of a plasma gradient, an absolute parametric instability is converted into a convective instability. Pesme, Laval, and Pellat[4] have stated that this only occurs if the size of the plasma is greater than some critical length $\gamma_0/\kappa'(v_1v_2)^{1/2}$. Below this critical length but above the basic gain length $(v_1v_2)^{1/2}/\gamma_0$, absolute instability was said to occur. We show here instead that, provided $\gamma_0^2/\kappa' v_1 v_2 > 1$ and the damping rates on the waves are small, absolute instability occurs for any size plasma greater than the basic gain length. Thus we conclude that the assumptions of Refs. 2 and 3 are unrealistic and that the results of Ref. 4 for a sharply bounded plasma are incorrect.

We begin, as usual, with the general three-wave inhomogeneous parametric-instability equations[2]:

$$(p+\gamma_1)a_1 + v_1\,\partial a_1/\partial x = \gamma_0 a_2^* \exp(i\textstyle\int \kappa\, dx), \qquad (1)$$

$$(p+\gamma_2)a_2^* - v_2\,\partial a_2^*/\partial x = \gamma_0 a_1 \exp(-i\textstyle\int \kappa\, dx), \qquad (2)$$

where a_1 and a_2 are the complex amplitudes of the two coupled waves, γ_1 and γ_2 are the respective damping rates, and v_1 and v_2 are the respective components of the group velocity along the density gradient and are assumed to be in opposite directions. The two waves are coupled by the pump wave to give an infinite homogeneous growth rate γ_0. The parameter $\kappa = k_0(x) - k_1(x) - k_2(x)$ is the spatial mismatch in wave number due to plasma gradients. We consider the case of a linearly varying κ with $\kappa = 0$ at the center of a plasma of length L. We also assume the behavior in time of $\exp(pt)$ so that we may eliminate the time derivatives in Eqs. (1) and (2). The growth rate γ_0 is assumed to be zero for $x < -L/2$ and $x > L/2$ and the equations are solved with the boundary conditions of fixed incoming waves, i.e., $a_1(-L/2) = a$ and $a_2(L/2) = b$. The value of p is adjusted so that these boundary conditions may be satisfied for other than trivial solutions. If solutions are found for $\mathrm{Re}\,p > 0$, then there is absolute instability.

An infinite, inhomogeneous plasma with a linear mismatch $\kappa(x) = \kappa' x$ is certainly not physically meaningful as $|x| \to \infty$. In Refs. 2 and 3 boundary conditions at $|x| = \infty$ were imposed leading to the singular result that the absolute instability which exists in an homogeneous system ($\kappa' = 0$) becomes a convective instability for an arbitrarily small but nonzero κ'. As we will show, the imposition of boundary conditions at finite points in the plasma preserves the absolute instability whose growth rate is then a *continuous* function of κ'.

The effective length of the system, insofar as the parametric instabilities in laser-produced plasmas are concerned, is determined by the x dependence of the parameters in Eqs. (1) and (2). In this paper we will assume that the growth rate $\gamma_0 = \gamma_0(x)$ controls the scale of the problem. In the case of Brillouin backscatter, the homogeneous-plasma growth rate[6] is $\gamma_0 \propto n_e^2 k_0$, where n_e is the electron density and k_0 is the pump wave number. In the WKB approximation of Eqs. (1) and (2), these parameters are replaced by their local values as functions of x. Thus $\gamma_0(x)$ approaches zero

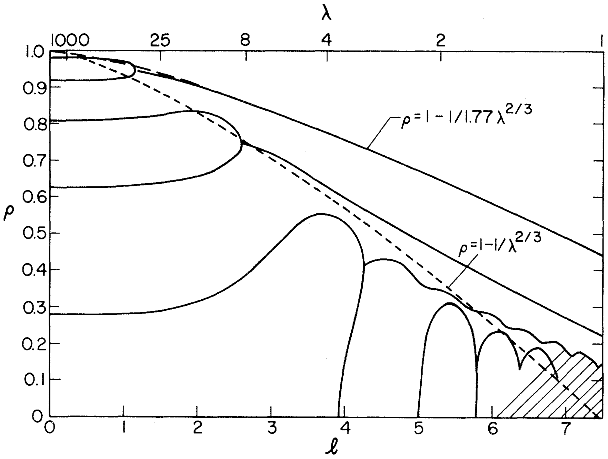

FIG. 1. Normal-mode growth rates as a function of l for $\lambda^{1/2} l = 7.5$. For reference two curves of growth rate as a function of λ are shown. The upper curve is indistinguishable from the numerical curve over most of the parameter range. The shaded region contains many roots which are not shown for clarity.

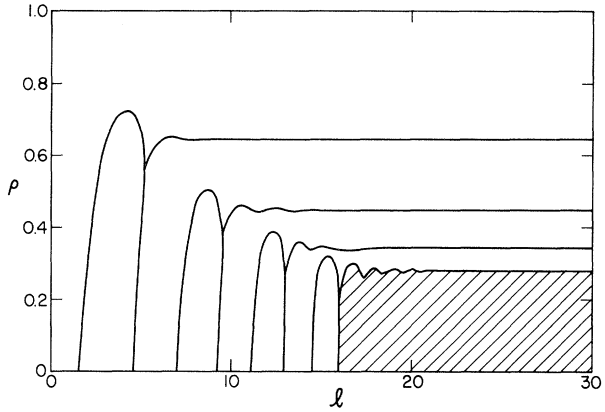

FIG. 2. Normal-mode growth rates as a function of l for fixed $\lambda = 2$. The shaded region has many roots which are not shown.

at the zero-density front of the plasma and at the reflection point $[\omega_0 = \omega_{pe}(x)]$ where $k_0 = 0$. Similar considerations apply to the γ_0 for Raman backscatter[6] where γ_0 vanishes at $\omega_0 = 2\omega_{pe}(x)$ and at the zero-density front.

We first consider the case in which $\gamma_0(x)$ is a step function,

$$\gamma_0(x) = \begin{cases} \gamma_0, & -L/2 < x < L/2, \\ 0, & \text{otherwise}, \end{cases}$$

and apply boundary conditions at $x = \pm L/2$. For a homogeneous plasma this reduces to the problem considered by Kroll[8] and others. In Figs. 1 and 2 results are presented for an inhomogeneous plasma as obtained by a combination of numerical and analytic techniques described below.

Numerical solutions to Eqs. (1) and (2) are obtained by setting $a_2(-L/2) = 0$ and $a_1(-L/2) = \text{const}$, integrating the equations to $x = L/2$, and adjusting the complex value of p so that the complex value of $a_2(L/2)$ is equal to zero.

A semianalytic analysis of these equations using the WKB approximation was also used. The substitution

$$a_1 = \psi \exp\left[\frac{1}{2}\left(\frac{p+\gamma_2}{v_2} - \frac{p+\gamma_1}{v_1}\right)x + \frac{i\kappa' x^2}{4}\right] \tag{3}$$

reduces Eqs. (1) and (2) to the form

$$\frac{d^2\psi}{dy^2} + q^2(y, \mu, \lambda)\psi = 0, \qquad q^2 = \tfrac{1}{4}(y - i\mu)^2 + \lambda + \tfrac{1}{2}i, \tag{4}$$

with boundary conditions $\psi = a$ at $y = -l/2$ and $[\frac{1}{2}(\mu + iy) + \partial/\partial y]\psi = b$ at $y = l/2$, where a and b are arbitrary constants and where we have defined the following dimensionless variables:

$$y = \sqrt{\kappa'}\,x, \quad \mu = \frac{1}{\sqrt{\kappa'}}\left(\frac{p+\gamma_1}{v_1} + \frac{p+\gamma_2}{v_2}\right), \quad \lambda = \gamma_0^2/\kappa' v_1 v_2, \quad l = \sqrt{\kappa'}\,L. \tag{5}$$

In the WKB approximation this leads to the dispersion relation

$$[\mu + i\tfrac{1}{2}l + 2\nu_r(\tfrac{1}{2}l)]\tan[\nu_i(\tfrac{1}{2}l) - \nu_i(-\tfrac{1}{2}l)] = -2\nu_i'(\tfrac{1}{2}l), \tag{6}$$

where the WKB solutions of Eq. (4) are $\psi_\pm(y) = \exp[\nu_r(y) \pm i\nu_i(y)]$ with[9]

$$\nu_r(y) = -\tfrac{1}{2}\ln q + \dots, \qquad \nu_i(y) = \int_0^y dy'\, q(y') + \dots. \tag{7}$$

The roots of the dispersion relation of Eq. (6) were found on a computer up to sixth order in the WKB approximation, allowing us to examine the convergence of the procedure directly.

The usual WKB validity condition $q' \ll q^2$ can be cast in the form

$$|\tfrac{1}{2}(y - 2i\rho\lambda^{1/2})| \ll 2|\tfrac{1}{4}(y - 2i\rho\lambda^{1/2})^2 + \lambda + \tfrac{1}{2}i|^{3/2},$$

where $\rho = \mu/2\lambda^{1/2}$ is the growth rate in units of the homogeneous-plasma absolute growth rate.[8] If $\lambda \gg 1$ and y has its largest value $\sqrt{\kappa'}L/2$, it is easy to see that the inequality is automatically satisfied if $\sqrt{\kappa'}L/2 \gg \lambda^{1/2}$. If $y \ll \lambda^{1/2}$, on the other hand, the inequality reduces to $\rho \ll 2\lambda(1-\rho^2)^{3/2}$. If we equate both sides of this inequality, we obtain the growth rate at the point of failure of the

WKB solution. For $\lambda \gg 1$ this condition yields $\rho = 1 - (2\lambda)^{-2/3}$. In Fig. 1 the curve $\rho = 1 - \lambda^{-2/3}$ is shown to be very close to the points of coalescence of the roots. We also see that the point of breakdown of the WKB solution defined by

$$\rho = 1 - 1/\sqrt{\pi}\lambda^{2/3} \tag{8}$$

describes the maximum growth rate of the absolute instability very accurately. This formula predicts that ρ is independent of L if $\lambda \gg 1$ which agrees with Fig. 2 for large L.

Below the coalescence points $\mathrm{Im}p$ is very nearly zero. Above the coalescence points the numerical solution shows that the two roots have the same $\mathrm{Re}p$ but oppositely signed $\mathrm{Im}p$, i.e., are complex conjugates. $|\mathrm{Im}p|$ increases approximately as $l/\lambda^{1/2}$, is nearly independent of κ', and increases linearly with L. Thus the most favorably matched point moves away from the center of the region where $\kappa = 0$. This point, however, always remains far from the boundary of the plasma so that the solutions are not inconsistent with the original WKB approximation.

In Fig. 1 we can trace the absolute-instability growth rate for various modes from their $\kappa' = 0$ ($\lambda \to \infty$) values continuously as functions of l (or equivalently κ' since $\lambda^{1/2}l = 7.5$) to beyond the points where the various roots pairwise coalesce. In addition, new unstable modes appear which are not unstable for $\kappa' = 0$. These also eventually coalesce with other roots as κ' increases. These new modes arise from the intersection of roots which are stable for small κ' and which have large $\mathrm{Im}p$ of equal magnitude and opposite sign. Beyond the point of intersection, the values of $\mathrm{Re}p$ take on opposite signs (stable and unstable) and $\mathrm{Im}p$ is very small. Apparently the distortion of the wave function by the mismatch permits additional unstable modes to satisfy the boundary conditions.

In summary, we have found that for a step-function γ_0, the absolute-instability growth rate goes to zero only for $\lambda < 1$, and for $\lambda = \mathrm{const} > 1$, $\mathrm{Re}p \to \mathrm{const}$ as $L \to \infty$. Therefore, the absolute instability exists for $\lambda > 1$ and arbitrary plasma length. The step-function model is, of course, unrealistic in detail but has the virtue of showing that the boundary conditions imposed far from the matching point have an important effect on the nature of the instability.

To determine to what extent the absolute character of the instability is caused by the discontinuous nature of γ_0 considered above, various smoothly varying $\gamma_0(x)$ were investigated numerically. Two cases were considered in which the gradients in γ_0 were made independent of the length of the system:

$$\gamma_0(x) = \begin{cases} 0, & x < -\frac{1}{2}L,\ x > \frac{1}{2}L, \\ \gamma_0(x \pm \frac{1}{2}L)/L_2, & x \pm \frac{1}{2}L < L_2, \\ \gamma_0, & x \pm \frac{1}{2}L > L_2; \end{cases}$$

and $\gamma_0(x) = \gamma_0 \tanh[(x \pm L/4)/L_2]$, where L_2 is the size of the region of large gradients in γ_0. We find that for both cases, if $L_2 \ll L$, [even for $L_2 > (v_1 v_2)^{1/2}/\gamma_0$], the results are essentially unchanged from those of Figs. 1 and 2. That is, the absolute instability discussed above is not a result of the discontinuous change in γ_0. As long as the effective matching point implied by the large $|\mathrm{Im}p|$ is not in a region of strong gradients in γ_0, $|\mathrm{Im}p|$ is unchanged. As L_2 is increased toward L, however, the effective matching point is pushed toward the center. If L is greater than the length for coalescence, the absolute instability goes away when the matching point reaches the center. This behavior is consistent with that found for the function $\gamma_0(x) = \gamma_0 \sin^2[\pi(x + \frac{1}{2}L)/L]$. In this case the sequence of unstable roots in the homogeneous plasma is distorted downward by the increasing inhomogeneity (or decreasing λ) rather than upward as in Fig. 1. They generally coalesce near $\rho = 0$ just before the absolute instability is completely suppressed. This is consistent with the behavior described above for the profiles with more uniform $\gamma_0(x)$. Large gradients in γ_0 appear to suppress the movement outward from the center of the matching points.

We have also found that if κ' is chosen to be a function of x such that λ is a constant in space, absolute instability can occur even for $\gamma_0(x) \propto \sin^2[\pi(x + \frac{1}{2}L)/L]$. This does not correspond to the case $\kappa = \kappa(0)x^2$ considered by Liu, Rosenbluth, and White[10] since the coupling is zero in the region where $\kappa' = 0$. All this goes to show that one cannot make general conclusions about the existence or nonexistence of absolute instability. One must consider all the inhomogeneous aspects of each problem.

The presence of damping (linear or nonlinear) can, of course, keep the system below the absolute-instability threshold. From Eq. (8) the explicit formula for the growth rate of the strongest growing mode is

$$\mathrm{Re}p = \left[\frac{2\gamma_0}{(v_1 v_2)^{1/2}}\left(1 - \frac{1}{\sqrt{\pi}\lambda^{2/3}}\right) - \left(\frac{\gamma_1}{v_1} + \frac{\gamma_2}{v_2}\right)\right] \times \frac{v_1 v_2}{v_1 + v_2}, \tag{9}$$

which is the same as the fastest growth rate for the homogeneous bounded system except for the factor $1 - 1/\sqrt{\pi}\lambda^{2/3}$. When damping is strong enough to make $\mathrm{Re}p < 0$, the finite size of the plasma may also play a role in the properties of the convective evolution of the instability as previously noted by several authors.[4,11]

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Ponderomotive-Force Effects in a Nonuniform Plasma*

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The effect of the ponderomotive force in the interaction of a capacitor rf field with a nonuniform plasma is investigated.

The interaction of intense radiation with nonuniform plasmas is presently a subject of great fundamental and practical interest. Recent experiments[1] have indicated that one of the most significant effects in this problem is the nonlinear modification of the density profile by the large electrostatic fields that arise due to linear mode conversion. In this Letter we present a simple time-dependent model which permits the isolation of the fundamental aspects associated with the density changes caused by the ponderomotive force.

We consider a plasma whose zeroth-order spatially dependent density is given by $n(x) = n_0(1 + x/L)$, in which L is the profile length scale and n_0 is the density at the spatial point $x = 0$. This plasma is assumed to be driven by external capacitor plates that generate a vacuum rf field given by $E_0 \exp(-i\omega_0 t)$. This highly simplified geometry has been used successfully[1] to make delicate measurements of the enhanced electric field in the plasma and the associated nonlinearities. The capacitor pump field generates a self-consistent electric field in the plasma which can be represented by $E(x,t)\exp(-i\omega_0 t)$. In the present work we assume that the time dependence of $E(x,t)$ is slow compared to the $2\pi/\omega_0$ time scale. Since we have in mind effects that occur in the neighborhood of the resonance point where ω_0 matches ω_p (the local electron plasma frequency), the effect of wave-particle interactions is neglected. This assumption is strictly correct for small pump amplitudes; at large amplitudes the spatially localized electric fields generated may interact strongly[2] with the particles. However, since we are interested in the initial stages of formation of these electric field spikes, the secondary effect of their interaction with the particles is neglected. The basic equation that describes the slow time behavior of this system can be obtained by combining the high-frequency response of the electron fluid together with Poisson's equation.