paper of Ref. 4, instead of that of Ref. 2. This *does* include column B of the table, and *does not* include column C yet.

[7]G. A. Rinker, Jr., and L. Wilets, Phys. Rev. Lett. 31, 1559 (1973). The author thanks Professor H. L. Anderson for informing him of this paper before it was published.

[8]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).

[9]We include $l=2$ levels for Ba^{56}, because R/r is small enough. Wave functions for extended-nucleus Coulomb potential should, in principle, be used, though here in fact those for pointlike ones are used; it makes practically no difference for large l.

[10]L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. Lett. 32, 562 (1974) (this issue).

Analytic Calculation to all Orders in $Z\alpha$ of Nuclear Size Effects in Vacuum Polarization*

Lowell S. Brown, Robert N. Cahn, and Larry D. McLerran
Physics Department, University of Washington, Seattle, Washington 98195
(Received 7 January 1974)

We have calculated the change in the vacuum polarization for muonic atoms with a high-Z nucleus arising from the finite extent of its charge distribution. In the calculation, the electron mass is set equal to zero and only the first term in an expansion in (nuclear radius)/(radius of the muonic state) is retained. The calculation is done to all orders in $Z\alpha$ and a simple closed-form result is obtained.

Muonic energy levels in high-Z atoms provide excellent tests of the validity of quantum electrodynamics. Many high-order Feynman diagrams contribute significantly because they enter with a factor $(Z\alpha)^n$ rather than just α^n. X-ray transitions in muonic atoms are sensitive to such higher-order corrections and accordingly have received extensive experimental attention. There appear to be discrepancies between the experimental results and the theoretical predictions.[1-4] These theoretical predictions take into account a variety of effects which perturb the basic energy levels given by the Dirac equations for a static pointlike source.[4] We are concerned here with only one of these effects: the perturbation of the vacuum polarization around the nucleus due to its finite extent.

The most interesting x-rays for testing quantum electrodynamics (QED) are those arising from simple systems where the muon moves in orbits which are neither too close to the nucleus (where the Lamb shift and the complications of nuclear physics would enter) nor too far from the nucleus (where the screening by electrons becomes too important). These constraints are satisfied by many muonic x-ray transitions, such as the $5g_{9/2}\to 4f_{7/2}$ transition in ^{208}Pb. In this system, the nuclear radius is about 6 fm, the muonic orbits have radii of 50–80 fm, and the first electronic Bohr orbit occurs at about 600 fm. For high angular momentum states, the muon spends very little time inside the nucleus.

Under these conditions, the muon sees a pure Coulomb potential together with the potential associated with the vacuum polarization from virtual electron-positron pairs. This vacuum-polarization potential is substantial in high-Z atoms, contributing about 2000 eV to the $5g_{9/2}\to 4f_{7/2}$ muonic transition in ^{208}Pb. The range of this potential is characterized by the Compton wavelength of the electron, ~400 fm, which is large compared to the radii ($r_{\rm Bohr}$) of the muonic or-

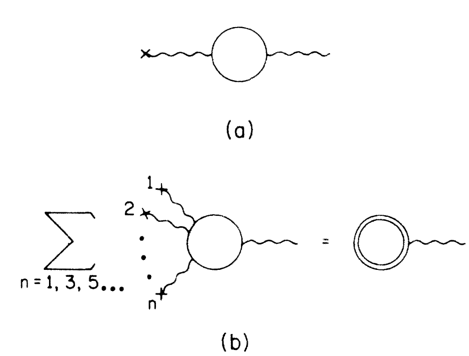

FIG. 1. (a) A representation of the Uehling potential. The nucleus is represented by ×. (b) A representation of all vacuum polarization effects of order $\alpha(Z\alpha)^n$, $n=1,3,5,\ldots$. If × represents a point source, the potential is the one discussed in Ref. 7. Our calculation corresponds to the difference between having × represent a point source and having it represent a nucleus of finite extent.

bits, which in turn are much larger than the nuclear radius (b). This justifies our fundamental approximations: setting the electron mass equal to zero and taking only the first term in an expansion in $b/r_{\rm Bohr}$. The validity of these approximations can be verified directly for the term of order $\alpha(Z\alpha)$ as we show below.

The first term in the potential due to vacuum polarization is the Uehling potential.[5] It is represented diagrammatically in Fig. 1. The diagram corresponds formally to a divergent integral. After charge renormalization there remains a finite electrostatic potential. The potential energy of a μ^- in this field is

$$V_{\rm Ueh}(r)=-\frac{\alpha Z\alpha}{\pi r}\int_{2m_e}^{\infty}d\kappa\, e^{-\kappa r}\left(\frac{2}{3\kappa^2}+\frac{4m_e^{\,2}}{3\kappa^4}\right)(\kappa^2-4m_e^{\,2})^{1/2}. \quad (1)$$

This result holds for a point source. If the source is of finite extent, we need only integrate this result over the source distribution. The change in the potential energy due to the finite extent is given by the expansion[4]

$$\delta V_{\rm Ueh}(r)=\frac{\alpha Z\alpha}{\pi}\left\{-\tfrac{1}{9}\langle r^2\rangle_{\rm nuc}\frac{1}{r^3}+\tfrac{1}{3}\langle r^2\rangle_{\rm nuc}\frac{m_e^{\,2}}{r}-\tfrac{1}{30}\langle r^4\rangle_{\rm nuc}\frac{1}{r^5}+\cdots\right\}, \quad (2)$$

where $\langle\cdots\rangle_{\rm nuc}$ denotes a mean value taken over the charge distribution of the nucleus.

The Uehling potential is just the first in a series of terms in which the electron loop is attached by photon propagators $2n+1$ times to the nucleus [see Fig. 1(b)]. The sum of all such diagrams gives all the terms in the vacuum polarization potential of order $\alpha(Z\alpha)^{2n+1}$. (Diagrams with an odd number of external photons vanish by Furry's theorem.)[6] If the source is pointlike, this potential is closely related to the electron Green's function in a Coulomb field. By constructing explicitly the Coulomb Green's function, Wichmann and Kroll[7] were able to represent the vacuum potential around a point source to all orders in $Z\alpha$.

We have calculated the difference between the full result when the external source has finite extent and the result when the source is pointlike. Equivalently, we have calculated the finite-size correction to the Wichmann-Kroll result. By setting the mass of the electron equal to zero, and calculating only the first term in $b/r_{\rm Bohr}$, we are restricting ourselves to the term analogous to the first term (proportional to $\langle r^2\rangle_{\rm nuc}/r$) in Eq. (2). We discuss below the validity of this approximation.

The details of our calculation are lengthy and not entirely trivial. We shall publish them in a longer paper. Here we only sketch the procedure we have followed. The vacuum-polarization charge density is given by the electron's Green's function with common space-time coordinates. For the point-source case this requires knowledge of the Coulomb Green's function. In our case, we need the Green's function for a potential which is Coulomb outside the nucleus, but which deviates from a Coulomb potential inside the nucleus. It suffices to know the charge density outside the nucleus since we know that charge is conserved and thus the charge inside the nucleus compensates for the charge induced outside. How the vacuum-polarization charge density is distributed inside the nucleus affects the potential only inside the nucleus. This is inconsequential for the high angular-momentum states of the muon which concern us. Only the lowest partial waves of the Green's function are influenced to leading order by the finite extent of the nucleus. In this way we have reduced the problem to that of constructing the lowest partial-wave Green's function outside the nucleus. Since the regular and irregular solutions to the Dirac equation outside the nucleus ($r>b$, where b is the joining radius) must be linear combinations of the solutions to the point-source case, the determination of this Green's function is possible if the regular solution is known at the surface of the nucleus. The procedure is quite analogous to that used in nonrelativistic quantum mechanics to determine scattering by a hard sphere.

The equal-time Green's function can be explicitly represented in terms of an infinite integral running along the imaginary energy axis. Because we have set the mass of the electron to zero, the integrand is tractable, involving squares of Whittaker functions for fixed parameters. The result is

$$\delta V_{\rm WK}(r)=\frac{\alpha}{\pi r}\left(\frac{b}{r}\right)^{2\lambda}\left[\frac{R-Z\alpha/(1+\lambda)}{1-Z\alpha R/(1+\lambda)}\right]\frac{\Gamma(4\lambda)|\Gamma(\lambda+iZ\alpha)|^4}{\lambda^2(2\lambda+1)^2\Gamma(2\lambda)^4}, \quad (3)$$

where $\lambda = [1-(Z\alpha)^2]^{1/2}$. Here R is the ratio, G/F, of the lower to the upper (small to the large) components of the Dirac equation for a regular $j=\frac{1}{2}$ solution evaluated for zero energy at the nuclear surface. Thus R can be determined by a simple numerical integration of the Dirac equation for $m_e=0$ and $E=0$, if the experimental nuclear charge density is available. In practice we have found it convenient to use a model for the electrostatic potential of the nucleus: $V=A(r^2+a^2)^{-1/2}$. This potential is joined continuously to a pure Coulomb potential for $r>b$. This model is solvable. The potential is compared to the experimental potential for Pb in Fig. 2.

As a check on our calculation, we have also computed the ratio R to lowest order in $Z\alpha$,

$$R=\tfrac{1}{2}Z\alpha(1-\tfrac{1}{3}\langle r^2\rangle_{\text{nuc}}/b^2). \quad (4)$$

Inserting this into Eq. (3) and expanding out the lowest order terms, we obtain the result to order $\alpha(Z\alpha)$. This is nothing but the finite-size correction to the Uehling potential and agrees, as it must, with the first term of Eq. (2).

The energy shift due to the potential in Eq. (3) may be calculated by evaluating its expectation value for a Dirac wave function. The results for various states are shown in Table I. Also shown are corrections due to finite-size effects for terms of order $\alpha(Z\alpha)$, i.e., the finite-size correction to the Uehling potential calculated from Eq. (2). The qualitative features are as follows: only for very high values of $Z\alpha$ (e.g., Pb) is it necessary to go beyond the Uehling result; the corrections due to the finite electron mass and to higher orders in b/r_{Bohr} are small fractions of the finite-size correction for the Uehling potential. This supports the assumptions we have made in our calculation to all orders in $(Z\alpha)$. Lead furnishes the most interesting case. There, for the $4f_{7/2}$ state, we find the energy is lowered by about 20 eV, while for the $5g_{9/2}$ state, it is lowered by about 6 eV. The Uehling piece of the finite-size correction accounts for 12 and 3 eV, respectively. Altogether, terms of order $\alpha(Z\alpha)^3$ and higher in the nuclear-size correction result in increasing the energy of the x ray associated with $5g_{9/2}\to 4f_{7/2}$ by 5 eV. (Essentially identical results are obtained for the transition $5g_{7/2}\to 4f_{5/2}$.)

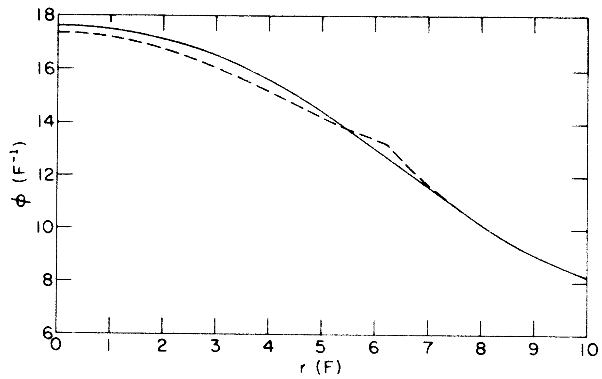

FIG. 2. A comparison of the model potential (dashed line), $V=A(r^2+a^2)^{-1/2}$ $(r<b)$, $V\propto 1/r$ $(r>b)$, with the potential for Pb derived from its experimental charge distribution (solid line) (Ref. 9).

This result has the same sign but is only one third the size of that given by the computer calculation of Rinker and Wilets.[8] The discrepancy between theory and experiment quoted by Rinker and Wilets for this transition is only slightly re-

TABLE. I. Finite-size corrections to vacuum polarization shifts of atomic energy levels, in eV.

		Order $\alpha(Z\alpha)^n$, $n=1,3,5,\ldots$; $m_e=0$, lowest order in b/r_{Bohr}	Order $\alpha(Z\alpha)$ only: $m_e=0$, lowest in b/r_{Bohr}	$m_e\neq 0$ correction	Higher order in b/r_{Bohr}
$^{40}Ca_{20}$	$3d_{5/2}$	−0.11	−0.10	0.02	0.00
	$4f_{7/2}$	−0.02	−0.02	0.01	0.00
$^{136}Ba_{56}$	$3d_{5/2}$	−15.9	−13.2	0.24	−0.35
	$4f_{7/2}$	−2.7	−2.0	0.14	−0.01
	$5g_{9/2}$	−0.7	−0.5	0.09	0.00
$^{208}Pb_{82}$	$3d_{5/2}$	−104	−83	0.69	−6.48
	$4f_{7/2}$	−20	−12	0.38	−0.15
	$5g_{9/2}$	−6	−3	0.25	−0.01

duced by substituting our result for finite-size corrections to terms of order $\alpha(Z\alpha)^3$ and higher. Typical results for the energy of the x ray are $E(\text{theor.}) - E(\text{expt.}) = 46 \pm 18$ eV for $5g_{9/2} \rightarrow 4f_{7/2}$ and 61 ± 21 eV for $5g_{7/2} \rightarrow 4f_{5/2}$ in ^{208}Pb.

The techniques employed in this calculation, especially taking the limit $m_e = 0$, may have a wider range of applicability in doing higher-order QED calculations in atomic systems. In particular, certain portions of the Wichmann-Kroll calculation can be simplified in this manner.

We wish to thank our colleague, Dr. L. Wilets, for bringing this problem to our attention and for discussions of the physics of the problem. One of us (R.N.C.) would like to thank Dr. S. J. Brodsky, Dr. P. Mohr, and Dr. E. Wichmann for useful discussions.

Note added.—After this manuscript was completed, we learned of a recent unpublished calculation by J. Arafune who computed the $\alpha(Z\alpha)^3$ term analogous to Eq. (3) for a model nucleus with uniform charge density. His numerical result for Pb agrees very well with ours.[10]

*Work supported in part by the U. S. Atomic Energy Commission.

[1]M. S. Dixit *et al.*, Phys. Rev. Lett. 27, 878 (1971).
[2]G. Backenstoss *et al.*, Phys. Lett. 31B, 233 (1970).
[3]H. K. Walter *et al.*, Phys. Lett. 40B, 197 (1972).
[4]See for example, J. Blomqvist, Nucl. Phys. B48, 95 (1972).
[5]E. A. Uehling, Phys. Rev. 48, 55 (1935).
[6]W. Furry, Phys. Rev. 51, 125 (1937).
[7]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).
[8]G. A. Rinker, Jr., and L. Wilets, Phys. Rev. Lett. 31, 1559 (1973).
[9]J. Heisenberg *et al.*, Phys. Rev. Lett. 23, 1402 (1969).
[10]J. Arafune, Phys. Rev. Lett. 32, 560 (1974) (this issue).

Continuous Spins in the Bondi-Metzner-Sachs Group of Asymptotic Symmetry in General Relativity

L. Girardello*
International Centre for Theoretical Physics, Trieste, Italy

and

G. Parravicini
Istituto di Fisica dell'Università, Milano, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Italy
(Received 28 August 1973)

It has been suggested that elementary particles should be associated with the irreducible representations of the Bondi-Metzner-Sachs group, on the grounds that in these, only discrete spins appear. We emphasize that the last statement strongly depends on the topology originally chosen for supertranslations, and we discuss in some detail how continuous spins appear when other equally reasonable topologies are chosen, which depend in a more explicit way on the smoothness of supertranslations.

The possibility of replacing the Poincaré group with the Bondi-Metzner-Sachs group[1] (herein denoted by B) for the space-time properties of elementary particles has been considered by some authors,[2] and also in connection with O'Raifeartaigh's theorem. B is the semidirect product of SL(2, C) times the Abelian group of supertranslations, i.e., an infinite-dimensional real vector space Λ of suitable smooth functions $f(\theta, \varphi)$ on the sphere S^2. The action of SL(2, C) on these functions is such that there exists a connection[3] between the elements f of Λ and the elements φ of the carrier space of the $D_{2,2}$ representation of SL(2, C).[4]

The idea mentioned above has been advanced recently in a clear way by McCarthy[3] with the suggestion that an elementary particle should be associated with a unitary irreducible representation (UIR) of B. One of the main motivations for this suggestion stems from the fact that in this author's noteworthy study of the induced UIR of B only compact little groups appear, hence the only spins allowed are discrete in contrast to the Poincaré case. These results were obtained by