[16]M. G. Piepho and G. E. Walker, to be published.

[17]In this case, the (3, 3) resonance width comes out much too large, however. It seems difficult to fit the scattering volume, resonance position, and width simultaneously with a one-term, energy-independent, separable potential. The Chew-Low theory, on the other hand, fits these three quantities very well by virtue of its energy dependence.

Muonic Heavy-Atom Spectrum and Nuclear Size Effect on Vacuum Polarization*

Jiro Arafune†
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637
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Nuclear finite size correction to the vacuum polarization is examined and applied to muonic atom spectrum. An analytic expression for the potential due to this correction is given. It increases the present discrepancy by 5 eV in the $5g_{9/2}$-to-$4f_{7/2}$ transition in lead.

The recent precise measurements[1] of x-ray spectra of muonic atoms have stimulated re-examination of theoretical calculations. In spite of careful checks on such effects as higher order vacuum polarization,[2,3] electron screening,[4] and electromagnetic excitation of nucleus,[5] there still remains a sizable discrepancy[1-4] between experimental values and theoretical calculations. The experimental value is, for example,[1] 431 285 ± 17 eV for the $5g_{9/2}$-to-$4f_{7/2}$ transition in ^{208}Pb, while the theoretical one[6] is 431 327 ± 12 eV, giving a discrepancy 42 ± 20 eV. One effect which has not been examined in detail so far is the finite nuclear size correction[7] to the higher order vacuum polarization. We present in this paper an analytic expression for the potential due to this correction:

$$\delta V(r) = -\frac{\alpha(Z\alpha)}{15\pi}\left\{1 - C_1(Z\alpha)^2 - C_2(Z\alpha)^2\frac{R}{r}\right\} \times \frac{1}{r}\left(\frac{R^2}{r^2}\right)^{[1-(Z\alpha)^2]^{1/2}}, \qquad (1)$$

with

$$C_1 = -3 + \frac{\pi^2}{3} - \frac{349}{13860} = 0.265,$$

$$C_2 = \frac{865}{2016} = 0.429,$$

for $r > R$, where R is the nuclear radius, about $1.20A^{1/3}$ fm (A is the mass number).

We comment on this equation: (1) This includes the lowest order (in $Z\alpha$) correction given by Blomqvist[2] and it reduces exactly to the main part of his, if the higher order terms in $(Z\alpha)$ are neglected and R^2 is replaced with $\frac{5}{3}\langle r^2\rangle$ in Eq. (1). (2) We neglect the electron mass, because the error due to this approximation has turned out to be $O((m_e r)^2)$ and not $O(m_e r)$, thus very small. (3) We neglect such terms as R^4/r^5 or higher, because R^2/r^2 is less than $\frac{1}{20}$ for $l \geq 3$, in muonic atoms. (It should be noted that we have checked that our method gives the same coefficient of the R^4/r^5 term as in Ref. 2, to the lowest order in $Z\alpha$.) This approximation enables us to neglect those virtual electrons involved in vacuum polarization which have total angular momenta larger than $\frac{3}{2}$ (we shall discuss it later). (4) We neglect higher orders in $(Z\alpha)^2$ for the constants C_1 and C_2, but we do not neglect the higher orders for the power of the exponent of $(R^2/r^2)^{[1-(Z\alpha)^2]^{1/2}}$ because $(R^2/r^2)^{-1}$ is very large. (5) We restrict ourselves to the vacuum polarization at the outside of the nucleus, because the inside is not important to the muonic states with high angular momenta, say, $l \geq 3$.

Now let us sketch how to obtain this result. We follow Wichmann and Kroll's[8] method, where the vacuum polarization due to virtual electrons is expressed in terms of a contour integral (over complex electron energy plane) of Dirac wave functions with arbitrary energy eigenvalues. The difference between their calculation and ours is only in the form of the nuclear Coulomb potential. We take

$$V(r_e) = -Z\alpha/r_e \quad \text{for } r_e \geq R,$$
$$= -3(Z\alpha)/2R + (Z\alpha)r_e^2/2R^3 \quad \text{for } r_e < R. \qquad (2)$$

This corresponds to a charge density which is constant inside the nuclear radius R and zero outside. We believe that the present calculation is insensitive to this approximation. In general there are two independent solutions to the Dirac equation with energy E, total angular momentum j, and fixed parity. We take for these solutions $v^{(1)}$, which is regular at $r_e=0$, and $v^{(2)}$, which is regular at $r_e=\infty$. Let $w^{(1)}$ and $w^{(2)}$ be the corresponding solutions for pointlike-nucleus potential (or $R=0$). Obviously we can set

$$\begin{aligned} v^{(1)}(r_e,E) &= w^{(1)}(r_e,E)+\eta(E)w^{(2)}(r_e,E), \\ v^{(2)}(r_e,E) &= w^{(2)}(r_e,E), \end{aligned} \tag{3}$$

for $r_e>R$, with $\eta(E)$ being constant.

In order to obtain $\eta(E)$, we solve $v^{(1)}(r_e,E)$ for $r_e<R$ and use the continuity of the spinor wave function at $r_e=R$. Using the spinor wave functions for $w^{(1)}$ and $w^{(2)}$ given in Ref. 7, we find that $w^{(2)}(R,E)$ has a singular factor R^{-2S} relative to $w^{(1)}(R,E)$ for $R\to 0$ $[S=\{(j+\frac{1}{2})^2-(Z\alpha)^2\}^{1/2}]$. This fact leads to

$$\eta = O(R^{2S}), \text{ for } R\to 0. \tag{4}$$

Substituting Eq. (3) into Wichmann and Kroll's formula, we get the vacuum polarization. The first term of $v^{(1)}$ in Eq. (3) gives the pointlike-nucleus result, and the second term gives the finite nuclear size correction. The correction is, therefore, proportional to $\eta(E)$ and thus to R^{2S} for $R\to 0$. This is why we need electron wave functions only for $j=\frac{1}{2}$. (We have used $j=\frac{3}{2}$ as well in getting the R^4/r^5 term to the lowest order in order to check the validity of our method and have got the same answer as Ref. 2.) At this stage we easily find that the error due to the finiteness of electron mass starts with m_e^2 and not m_e. So it is safe to put $m_e=0$. Finally we get the expression (1). The energy shifts to several muonic atom levels are given in Table I.[9]

We see that the higher order correction has the sign to increase the present discrepancy,[1-4] and contributes by pretty small amounts to the energy differences though each shift itself is large. The discrepancy mentioned in the second paragraph is now increased to 47 ± 20 eV.

The author is very grateful to Professor Y. Nambu for bringing this problem to the author's attention, giving him very kind comments and useful references, and reading the manuscript. He is also thankful to Professor H. L. Anderson for interesting discussions.

After this calculation was performed, we received a report of Rinker and Wilets,[7] who show

TABLE I. Finite-size correction to muonic atoms. The expectation value of Eq. (1) is given in column A. This includes column B, the correction to the lowest order (Uehling) potential. The higher order correction is given in column C.

Atom	Level	(A) $Z\alpha+(Z\alpha)^3+\cdots$ (eV)	(B) $Z\alpha$ (eV)	(C) (A) − (B) (eV)
Ba^{56}	$3d_{3/2}$	−15.9	−13.6	−2.3
	$3d_{5/2}$	−15.2	−12.9	−2.3
	$4f_{5/2}$	− 2.6	− 2.0	−0.7
	$4f_{7/2}$	− 2.6	− 1.9	−0.7
	$5g_{7/2}$	− 0.7	− 0.5	−0.2
	$5g_{9/2}$	− 0.7	− 0.5	−0.2
Pb^{82}	$4f_{5/2}$	−20.3	−12.7	−7.6
	$4f_{7/2}$	−19.3	−11.7	−7.6
	$5g_{7/2}$	− 5.8	− 2.9	−2.9
	$5g_{9/2}$	− 5.7	− 2.8	−2.9
	$5g_{7/2}$-$4f_{5/2}$	14.5	9.8	4.7
	$5g_{9/2}$-$4f_{7/2}$	13.6	8.9	4.7

that the relevant correction is very large. Since they have adopted a quite different technical approach, making full use of computer and subtracting gauge-dependent terms and renormalization term by computer, the present author is not able to check their result, or to explain the discrepancy between the two, at present. Further examination of this problem is desirable.

Note added.—After submitting this article for publication, we received a report by Lowell S. Brown, Robert N. Cahn, and Larry D. McLerran; with a similar analytic calculation they independently obtained the relevant corrections in a very good agreement with ours.[10] We thank them for the correspondence.

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†On leave of absence from Institute for Nuclear Study, University of Tokyo, Tanashi, Tokyo, Japan.

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[2]J. Blomqvist, Nucl. Phys. B48, 95 (1972).

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[6]Here we have modified the estimation of Ref. 2, adopting the electron screening calculation of the first

paper of Ref. 4, instead of that of Ref. 2. This *does* include column B of the table, and *does not* include column C yet.

[7]G. A. Rinker, Jr., and L. Wilets, Phys. Rev. Lett. 31, 1559 (1973). The author thanks Professor H. L. Anderson for informing him of this paper before it was published.

[8]E. H. Wichmann and N. M. Kroll, Phys. Rev. 101, 843 (1956).

[9]We include $l=2$ levels for Ba^{56}, because R/r is small enough. Wave functions for extended-nucleus Coulomb potential should, in principle, be used, though here in fact those for pointlike ones are used; it makes practically no difference for large l.

[10]L. S. Brown, R. N. Cahn, and L. D. McLerran, Phys. Rev. Lett. 32, 562 (1974) (this issue).

Analytic Calculation to all Orders in $Z\alpha$ of Nuclear Size Effects in Vacuum Polarization*

Lowell S. Brown, Robert N. Cahn, and Larry D. McLerran
Physics Department, University of Washington, Seattle, Washington 98195
(Received 7 January 1974)

We have calculated the change in the vacuum polarization for muonic atoms with a high-Z nucleus arising from the finite extent of its charge distribution. In the calculation, the electron mass is set equal to zero and only the first term in an expansion in (nuclear radius)/(radius of the muonic state) is retained. The calculation is done to all orders in $Z\alpha$ and a simple closed-form result is obtained.

Muonic energy levels in high-Z atoms provide excellent tests of the validity of quantum electrodynamics. Many high-order Feynman diagrams contribute significantly because they enter with a factor $(Z\alpha)^n$ rather than just α^n. X-ray transitions in muonic atoms are sensitive to such higher-order corrections and accordingly have received extensive experimental attention. There appear to be discrepancies between the experimental results and the theoretical predictions.[1-4] These theoretical predictions take into account a variety of effects which perturb the basic energy levels given by the Dirac equations for a static pointlike source.[4] We are concerned here with only one of these effects: the perturbation of the vacuum polarization around the nucleus due to its finite extent.

The most interesting x-rays for testing quantum electrodynamics (QED) are those arising from simple systems where the muon moves in orbits which are neither too close to the nucleus (where the Lamb shift and the complications of nuclear physics would enter) nor too far from the nucleus (where the screening by electrons becomes too important). These constraints are satisfied by many muonic x-ray transitions, such as the $5g_{9/2} \to 4f_{7/2}$ transition in ^{208}Pb. In this system, the nuclear radius is about 6 fm, the muonic orbits have radii of 50–80 fm, and the first electronic Bohr orbit occurs at about 600 fm. For high angular momentum states, the muon spends very little time inside the nucleus.

Under these conditions, the muon sees a pure Coulomb potential together with the potential associated with the vacuum polarization from virtual electron-positron pairs. This vacuum-polarization potential is substantial in high-Z atoms, contributing about 2000 eV to the $5g_{9/2} \to 4f_{7/2}$ muonic transition in ^{208}Pb. The range of this potential is characterized by the Compton wavelength of the electron, ~400 fm, which is large compared to the radii (r_{Bohr}) of the muonic or-

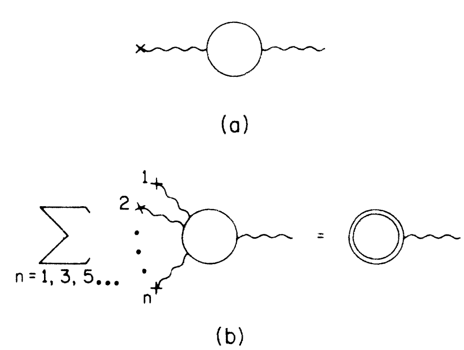

FIG. 1. (a) A representation of the Uehling potential. The nucleus is represented by ×. (b) A representation of all vacuum polarization effects of order $\alpha(Z\alpha)^n$, $n=1,3,5,\ldots$. If × represents a point source, the potential is the one discussed in Ref. 7. Our calculation corresponds to the difference between having × represent a point source and having it represent a nucleus of finite extent.