Partition Function for a Two-Dimensional Plasma in the Random-Phase Approximation*

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The partition function for a two-dimensional plasma is evaluated within the random-phase approximation. The periodic boundary conditions are taken fully into account by including the periodic image interactions. In the guiding-center limit, the "negative temperature" threshold energy is evaluated, and a value different from previous calculations results. When an identical random-phase evaluation is applied to the finite-gyroradius plasma, the Salzberg-Prager-May equation of state is recovered.

Considerable interest has arisen lately in the equilibrium statistical mechanics of two-dimensional plasmas, both in the "guiding center" and finite-gyroradius limits.[1-8] The "guiding center" model is particularly interesting because its total phase volume is finite, so that above a critical energy $\mathcal{E}_m$, the temperature is formally negative.[9] Here I evaluate the partition function for both systems within the random-phase approximation, and so arrive at the threshold energy $\mathcal{E}_m$.

In the random-phase approximation, periodic boundary conditions are implicit, which means that in calculating the total energy one must include the interactions of all the charges with the images of all the others. Until now this fact has not been appreciated. I use the two-body noncentral Ewald potential to calculate the energy of the system, which includes the image interactions and leads to a volume dependent term. For the finite-gyroradius case, the Salzberg-Prager-May equation of state is recovered, and for the guiding-center model a new value of $\mathcal{E}_m$ results.

I proceed from the canonical ensemble, which apparently has not been done directly for the guiding-center model. (For energies near $\mathcal{E}_m$, the usual steepest-descent evaluation of the partition function may not be assumed to imply the equivalence of the canonical and microcanonical ensembles.) All of my evaluations of thermodynamic quantities derive explicitly from the partition function.

The total energy for N positive and N negative charges in a box of volume V may be written as

$$\mathcal{E} = \sum_{i=1}^{2N} \vec{p}_i^{\,2}/2m_i + \sum_{i<j} \psi(\vec{x}_{ij}) + \mathcal{E}_0.$$

$\mathcal{E}_0$ is a constant which will be specified below. $\psi(\vec{x}_{ij})$ is the Ewald potential which includes the periodic images:

$$\psi(\vec{x}_{ij}) = (4\pi e_i e_j/lV)\sum_{\vec{k}}{}' k^{-2} \exp(i\vec{k}\cdot\vec{x}_{ij}).$$

$\vec{k} = 2\pi\vec{n}/V^{1/2}$; $\vec{n}$ is a vector with integer components; V is the volume of the system; and the prime on the summation means to omit $\vec{k}=0$. The ith charge is a very long rod of length l and charge e_i. Following Brush, Shlin, and Teller[10] and Nijboer and DeWette,[11] one may put $\psi(\vec{x}_{ij})$ into a form convenient for numerical evaluation,

$$\psi(\vec{\xi}_{ij}) = (e_i e_j/l)\{E_1(\pi\xi_{ij}^{\,2}) - 1 + \sum_{\vec{n}}{}'[\exp(-\pi n^2 + 2\pi i\vec{n}\cdot\vec{\xi}_{ij})/\pi n^2 + E_1(\pi|\vec{n}-\vec{\xi}_{ij}|^2)]\}, \tag{1}$$

where $\vec{\xi}_{ij} = \vec{x}_{ij}/V^{1/2}$, and $E_1(x)$ is the exponential integral. The constant $\mathcal{E}_0 = N\lim_{x\to 0}[\psi(x/V^{1/2}) - \varphi(x)]$, where $\varphi(x) = -(2e^2/l)\ln x$ is the two-body Coulomb interaction. The numerical value of $\mathcal{E}_0$ turns out to be $\mathcal{E}_0 = -2.62(Ne^2/l) + (Ne^2/l)\ln V$.

The partition function for the finite-gyroradius case is

$$Z = [V^{2N}/(N!)^2 h^{4N}]\int d^{6N}p\, d^{6N}\xi\, \exp(-\mathcal{E}/\theta),$$

where $\theta = k_B T$ is the temperature in energy units. The momentum-space part is trivial, and the configuration-space part becomes

$$Z_{\text{config}} = \int d^{6N}\xi \exp\{[\mathcal{E}_0 + \sum_{i<j}\psi(\vec{\xi}_{ij})]\theta^{-1}\}.$$

Following Taylor, let us invoke the random-phase approximation to convert the integral over the ξ's to one over density variables

$$r_{\vec{k}}^2=\sum_{i,j}(e_ie_j/l^2V^2)\exp[i\vec{k}\cdot(\vec{x}_i-\vec{x}_j)],$$

with corresponding Jacobian

$$J=\prod_{\vec{k}}{}'(V^2l^2/2Ne^2)\exp[-(V^2l^2/2Ne^2)r_{\vec{k}}^2].$$

This gives

$$Z_{\text{config}}=\int_0^\infty\prod_{\vec{k}}{}'(V^2l^2/2Ne^2)\exp[-(V^2l^2/2Ne^2)r_{\vec{k}}^2-(\mathcal{E}_0+2\pi Vr_{\vec{k}}^2/k^2)/\theta+4\pi Ne^2/l\theta Vk^2]\,dr_{\vec{k}}^2. \tag{2}$$

The integrations over $r_{\vec{k}}^2$ are easily done and give

$$Z_{\text{config}}=\prod_{\vec{k}}{}'(1+k_D^2/k^2)^{-1}\exp(k_D^2/k^2-\mathcal{E}_0/\theta),$$

where $k_D^2\equiv4\pi Ne^2/lV\theta$, which may be represented as an integral over k, to give ($m_\pm$ are the masses of the rods)

$$\begin{aligned}Z&=\frac{V^{2N}}{(N!)^2}\left(\frac{2\pi m_+\theta}{h^2}\right)^N\left(\frac{2\pi m_-\theta}{h^2}\right)^N\exp\left\{-\frac{\mathcal{E}_0}{\theta}-\frac{V}{2\pi}\int_{(4\pi/V)^{1/2}}^\infty k\,dk\left[\ln\left(1+\frac{k_D^2}{k^2}\right)-\frac{k_D^2}{k^2}\right]\right\}\\&=\frac{V^{2N}}{(N!)^2}\left(\frac{2\pi m_+\theta}{h^2}\right)^N\left(\frac{2\pi m_-\theta}{h^2}\right)^N\exp\left[-\frac{\mathcal{E}_0}{\theta}-\frac{Ne^2}{l\theta}+\left(1+\frac{Ne^2}{l\theta}\right)\ln\left(1+\frac{Ne^2}{l\theta}\right)\right].\end{aligned}\tag{3}$$

The various thermodynamic functions may be computed from Eq. (3). The pressure is $P=-\theta\partial(\ln Z)/\partial V=2(N/V)\theta(1-e^2/2l\theta)$, which is the Salzberg-Prager-May equation of state.[7,8] The internal energy is $\langle\mathcal{E}\rangle=\theta^2\partial(\ln Z)/\partial\theta=2N\theta+\mathcal{E}_0-(Ne^2/l)\ln(1+Ne^2/l\theta)$. The entropy is

$$S=k_B[\theta\partial(\ln Z)/\partial\theta+\ln Z]=2Nk_B\{\ln[2\pi(m_+m_-)^{1/2}\theta V/h^2N]+2\}+k_B[\ln(1+Ne^2/l\theta)-Ne^2/l\theta].$$

Note that the k integral in Eq. (3) has been cut off at the shortest wave number in the system, which also should have been done in the ring sum of Hauge and Hemmer.[3] Also note the combination of the logarithms gives an *extensive* thermodynamic limit for large N and V, when $\theta>0$.

Results for the "guiding center" model may be obtained by ignoring the momentum-space contribution to Z. Thus we find for the energy and entropy

$$\langle\mathcal{E}\rangle_{\text{g.c.}}=\mathcal{E}_0-(Ne^2/l)\ln(1+Ne^2/l\theta), \tag{4}$$

$$S_{\text{g.c.}}=k_B[\ln(1+Ne^2/l\theta)-Ne^2/l\theta]. \tag{5}$$

We may eliminate θ in favor of energy to obtain

$$S_{\text{g.c.}}=k_B\{1-(\langle\mathcal{E}\rangle_{\text{g.c.}}-\mathcal{E}_0)(Ne^2/l)^{-1}-\exp[-(\langle\mathcal{E}\rangle_{\text{g.c.}}-\mathcal{E}_0)(Ne^2/l)^{-1}]\}. \tag{6}$$

The temperature is given by $T^{-1}=\partial S/\partial\langle\mathcal{E}\rangle_{\text{g.c.}}$. Equations (4)–(6) hold for $\theta=k_BT>0$. Therefore, the threshold value of $\langle\mathcal{E}\rangle_{\text{g.c.}}$ may be obtained by letting $T\to+\infty$, and gives the threshold energy

$$\mathcal{E}_m=\mathcal{E}_0=-2.62(Ne^2/l)+(Ne^2/l)\ln N, \tag{7}$$

which differs from previous results.[5,8]

Previous evaluations of $\mathcal{E}_m$ have assumed an equivalence between the sum of the pairwise potentials and $\int\vec{E}^2d^3x/8\pi-\int\vec{E}_{\text{self}}^2d^3x/8\pi$, where $\vec{E}$ is the electric field expressed as a Fourier series, and $\int\vec{E}_{\text{self}}^2d^3x/8\pi$ is the infinite Coulomb self-energy of the charges. The Fourier representation of the $\vec{E}$, however, assumes *periodic* boundary conditions. Therefore, the two energies are equal only if the image charges are included in the sums of the pairwise interactions. The expression in Eq. (7) can be interpreted as the sum of the energies of the interaction of each of the charges with its own images.[12] Thus the threshold distribution is still the random distribution, as previously determined. A different conclusion was reached in Ref. (8).

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[12]If the interaction potential had been scaled so that $\varphi(x) = -(2e^2/l)\ln(x/l)$, then $\ln V$ in Eq. (7) would have been replaced by $\ln(V/l^2)$. This convention follows from developing the two-dimensional Coulomb interaction as a limit of the three-dimensional interaction of finite-length rods (H. E. DeWitt, private communication).

Measurements of Electron Density Evolution and Beam Self-Focusing in a Laser-Produced Plasma*

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Space- and time-resolved interferometric measurements of electron density in a CO_2-laser–produced plasma in helium show the development and evolution of radial profiles with on-axis minima, resulting in self-focusing of the laser beam.

The advent of high-power, long-wavelength CO_2 lasers has led to the possibility of achieving controlled thermonuclear fusion by laser heating of magnetically confined plasmas with densities near but below the critical density corresponding to the laser wavelength.[1] One of the fundamental requirements of this reactor scheme is the ability of the plasma to contain the laser beam in a long, thin, linear plasma column. For such containment, plasma refractive properties require that, transverse to the beam axis, the plasma density profile must have an on-axis minimum. We report in the present work measurements of the evolution of the electron-density profile in a laser-produced plasma in the absence of a magnetic field, demonstrating that electron-density profiles favorable for beam containment can be created by the action of the laser beam itself. In addition, beam self-focusing accompanying such favorable density profiles is observed to occur.

The plasma was produced at the focal spot of a TEA (transverse-excitation atmospheric) CO_2 laser with unstable resonator optics in a 5-m confocal cavity, which gives an annular output beam with 5-cm i.d. and 10-cm o.d., pulse energy up to ~30 J, pulse half-power width ~150 nsec, and beam divergence less than 1 mrad. The beam is focused by a KCl lens of 45 cm focal length. The radial intensity distribution at the focal plane is roughly Gaussian, with a measured focal-spot diameter of less than 1 mm, approximately equal to that expected from spherical aberration. The experiments were performed in helium gas at an initial pressure of 30 Torr, which would produce an electron density 2×10^{18} cm^{-3} for complete double ionization, corresponding to $\frac{1}{5}$ the critical density for the 10.6-μm incident CO_2 beam.

Typical plasma luminous images are shown in Fig. 1(a) by the three successive framing pictures, beginning at approximately 10 nsec after gas breakdown with 50 nsec delay between frames. (The incident beam propagates from left to right.) After breakdown the plasma size increases. At a given time, it is approximately cylindrical in shape with a bright shell, and its length along the beam is several times its diameter.

The electron-density measurements were made with a modified Mach-Zehnder interferometer with an internal focus, shown schematically in Fig. 1(b). The beam from a He-Ne laser is focused in the plasma by a lens outside the interferometer, and the recombined beam is monitored with a photomultiplier. One fringe shift corresponds to 3.5×10^{17} electrons/cm^2. When the interferometer is adjusted for uniform intensity across the recombined beam, a linear change of optical path in one arm of the interferometer gives a sinusoidal signal on the photomultiplier. The plasma can be scanned on a shot-to-shot basis by moving either the interferometer lens or