Plasma Heating by Alfvén-Wave Phase Mixing

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We propose the heating of collisionless plasma by utilizing a spatial phase mixing by shear Alfvén wave resonance and discuss the application to toroidal plasma. The phase mixing takes place as a result of the nonuniform Alfvén speed. The approximate heating rate per cycle of the wave frequency is given by $(b_0^2/\mu_0)\kappa/k_\perp$, where κ is the measure of the nonuniformity, $k_\perp$ is the wave number perpendicular to the direction of the magnetic field and the nonuniformity, and b_0 is the flux density of the applied-wave magnetic field.

Plasma heating is one of the most important issues in the success of controlled thermonuclear fusion. In particular, heating beyond the temperature achieved by Ohmic heating in a toroidal machine is a very crucial problem. In such a regime one promising way is to use electromagnetic waves. In this regime, heating by waves must rely on collisionless dissipation. Ion-cyclotron-resonance heating,[1] lower-hybrid-resonance heating,[2] and parametric excitation[3] are some of a few methods proposed so far. However, because of the relatively short wavelengths, these methods have intrinsic difficulty in coupling the wave energy to the plasma. We propose in this Letter the use of the resonance of a shear Alfvén wave that has a much longer wavelength.

In a nonuniform plasma, the Alfvén speed v_A is a function of position in the direction of the nonuniformity x. The shear Alfvén wave whose dispersion relation is given by $\omega = k_\parallel v_A$, where $k_\parallel$ is the wave number parallel to the magnetic field, meets the resonant condition at a local point (plane) x_0 in space where the excited frequency ω_0 satisfies $\omega_0 = k_\parallel v_A(x_0)$. Hence, if a surface magnetohydrodynamic (MHD) wave[4] is excited by an external coupler, the wave will be phase mixed by this resonance and its energy dissipated to the plasma. We use the surface wave rather than the magnetosonic wave as the coupling wave because the magnetosonic wave will propagate through the plasma and may produce undesirable effects. This may be achieved by choosing $k_\perp$, the wave number perpendicular to the density gradient as well as to the ambient magnetic field (k_θ in the cylindrical plasma), larger than $k_\parallel$ such that $k_\perp v_A > \omega_0$ is satisfied for the minimum Alfvén speed.

We assume a hot plasma in a straight but sheared magnetic field. For such a geometry the linearized ideal MHD equations give

$$\mu_0\rho_m\ddot{\vec{\xi}} - (\vec{B}\cdot\nabla)^2\vec{\xi} = -\mu_0\nabla\vec{p} - \vec{B}\vec{B}\cdot\nabla\nabla\cdot\vec{\xi}, \tag{1}$$

where $\vec{\xi}$ is the displacement vector, $\vec{p}$ the total pressure $(=p+\vec{b}\cdot\vec{B}/\mu_0)$, ρ_m the plasma mass density, $\vec{b}$ the perturbed magnetic field, and $\vec{B}$ the flux density of the ambient magnetic field,

$$\vec{B} = B_z(x)\vec{e}_z + B_y(x)\vec{e}_y. \tag{2}$$

(In a toroidal plasma, x, y, z correspond to radial, poloidal, and toroidal directions, respectively.) If we combine Eq. (1) with Maxwell's equations,

$$\vec{b} = \vec{B}\cdot\nabla\vec{\xi} - \vec{B}\nabla\cdot\vec{\xi} - (d\vec{B}/dx)\xi_x, \tag{3}$$

and the equation of state (after eliminating number density by using the continuity equation)

$$p = -\xi_x dP/dx - \gamma P\nabla\cdot\vec{\xi}, \tag{4}$$

P being the plasma static pressure, we obtain the following wave equation for ξ_x:

$$\frac{d}{dx}\left(\frac{\epsilon\alpha B^2}{\epsilon - \alpha B^2 k_\perp^2(x)}\frac{d\xi_x}{dx}\right) + \epsilon\xi_x = 0. \tag{5}$$

Here

$$\epsilon(x) = \omega^2\mu_0\rho_m(x) - k_\parallel^2(x)B^2(x),$$

$$\alpha(x) = 1 + \gamma\beta + \gamma^2\beta^2 k_\parallel^2 B^2/(\omega^2\mu_0\rho_m - \gamma\beta k_\parallel^2 B^2),$$

$$\beta(x) = \mu_0 P/B^2.$$

We assume $\beta \lesssim 1$, so that α is always positive. In the derivation of Eq. (5), we have assumed a perturbation of a form $\xi_x = \xi_x(x)\exp[i(k_z z + k_y y - \omega t)]$ with $\omega = \omega_0 + i\delta$ $(\delta = 0+)$ and have adopted local rectangular coordinates with $\vec{e}_\parallel = \vec{B}/B$ and $\vec{e}_\perp = \vec{e}_\parallel \times \vec{e}_x$. We note that near resonance, $\omega^2 = k_\parallel^2(x_0) \times v_A^2(x_0)$, where $\epsilon \simeq 0$, Eq. (5) reduces to

$$\frac{d^2\xi_x}{dx^2} + \left(\frac{d\ln\epsilon}{dx} - 2\frac{d\ln k_\perp}{dx}\right)\frac{d\xi_x}{dx} - k_\perp^2\xi_x = 0, \tag{5a}$$

which is singular at $x = x_0$. The existence of such a localized singularity in the MHD equations has been noticed by Pridmore-Brown for a cold plasma.[5] This singularity causes the phase mixing of the wave.[6] The solution of Eq. (4) which is singular at $x=x_0$ can be written near $x=x_0$ as

$$\xi_x = C\ln(x-x_0+i\delta'), \tag{6}$$

and the corresponding $\vec{e}_\perp$ component of the displacement is given by

$$\xi_\perp = (iC/k_\perp)(x-x_0+i\delta')^{-1}, \tag{7}$$

where $\delta' = \mathrm{Im}\epsilon/(d\epsilon/dx)$. The absorption rate of energy W by the plasma due to the phase mixing is obtained from

$$dW/dt = \mathrm{Re}\int \vec{J}\cdot\vec{E}^*\,dV, \tag{8}$$

in which $\vec{E} = i\omega\vec{\xi}\times\vec{B}$ and $\vec{J} = \nabla\times\vec{b}/\mu_0$. It turns out that only the compressional component of $\vec{b}$,

$$b_\parallel = -B(ik_\perp\xi_\perp + d\xi_x/dx) - (dB/dx)\xi_x,$$

contributes to Eq. (8). The major contribution to the integral (8) comes from near the singular point x_0. Hence, using Eqs. (6) and (7),

$$\frac{dW}{dt} \approx \pi L_y L_z \frac{\omega_0|C|^2}{\mu_0}\left(\frac{d\epsilon/dx}{\alpha k_\perp^2}\right)_{x_0}\frac{\gamma}{|\gamma|}$$
$$= \omega_0\pi L_y L_z \frac{|C|^2}{\mu_0}\left|\frac{d\epsilon/dx}{\alpha k_\perp^2}\right|_{x_0}, \tag{9}$$

where L_y and L_z are the size of the plasma in the y and z directions. The above result shows the existence of collisionless absorption of an applied field.

To evaluate the rate of power absorption quantitatively we have to obtain C, which is a measure of the plasma displacement caused by an applied high-frequency magnetic field. This can be done rigorously only when the geometric configuration of a plasma is specified. To obtain an approximate representation, we take again a slab geometry with linearly changing density and magnetic pressure, and solve the boundary-value problem. The plasma is taken to be semi-infinite ($x>0$) with its equilibrium density ρ_m and magnetic pressure $B^2/2\mu_0$ being constant except for $0 \le x \le a$, where $\rho_m(x) = \rho_0 x/a$ and $B^2(x) = B_0^2(1-\kappa_B x)$. The plasma temperature is uniform for all $x>0$. Thus, ϵ depends linearly on x within $0 \le x \le a$ and is constant otherwise. For simplicity we further assume $\beta \ll 1$, $B_z \gg B_y$, and $k_y \gg k_z$. With these assumptions, we then have $k_\perp \simeq k_y$, and $|\alpha B^2 k_\perp^2| \gg |\epsilon|$, and Eq. (5) can be approximated as

$$\frac{d^2\xi_x}{dx^2} + \frac{d\ln\epsilon}{dx}\frac{d\xi_x}{dx} - k_y^2\xi_x = 0. \tag{10}$$

The external driving source is represented by a sheet current in the vacuum,

$$\vec{J}_s(x,y,z,t) = \vec{I}_0\,\delta(x+h)\exp[i(k_y y + k_z z - \omega_0 t)].$$

The entire boundary-value problem then can be solved exactly using Eq. (10) and the familiar boundary conditions at $x=0$ and a.[7] The coupling is achieved by the $\vec{J}\times\vec{B}$ force which drives a magnetic compression in the plasma. The calculation is straightforward and the result is

$$Ck_\parallel B_0(x=0) = -2ib_s(x=0)/\Gamma, \tag{11}$$

where b_s is the x component of the wave magnetic flux density provided by the external circuit and

$$\Gamma = [I_0(g) - I_1(g)]\frac{K_1(d) - K_0(d)}{I_1(d) + I_0(d)} + K_0(g) + K_1(g),$$

with $g = -|k_y|x_0 < 0$ and $d = |k_y|(a-x_0)$. In the limit $|g|$, $|d| \gg 1$, we have

$$|\Gamma| \simeq (2\pi)^{1/2}e^{|g|}|g|^{-1/2},$$

and $|C|$ becomes for $\vec{I}_0 \simeq I_0\vec{e}_z$

$$|C| \simeq \left(\frac{2}{\pi}\right)^{1/2}\frac{|b_{s0}|}{|B_0 k_\parallel(x=0)|}(|k_y|x_0)^{1/2}. \tag{12}$$

Here $0 < x_0 < a$ and

$$|b_{s0}| \simeq \tfrac{1}{2}\mu_0 I_0\exp[-|k_y|(x_0+h)]. \tag{13}$$

From (9) and (11) we obtain the following estimate of the energy absorption rate:

$$\frac{dW}{dt} \simeq 2\omega_0(L_y L_z x_0)\frac{|b_{s0}|^2}{\mu_0}|k_y|^{-1}\left|\frac{d}{dx}\left(\frac{\omega_0^2}{k_\parallel^2 v_A^2}\right)\right|_{x_0}. \tag{14}$$

This expression can also be obtained from the evaluation of the Poynting vector.[7] Note that in the cylindrical coordinates $|b_{s0}|$ decays as r^{-m} (m is the azimuthal mode number) rather than exponentially.[8] Also note, because we use the cutoff mode with respect to the magnetosonic as well as electromagnetic waves, there is no radiation loss. Consequently, the heating efficiency is decided only by the Ohmic loss in the coil.

The above result can be compared with heating by transit-time damping,[9] which also uses Alfvén waves. Aside from numerical factors, the above expression gives a heating rate larger than that by transit-time damping by a factor of $1/\beta$.

For practical applications we must note the following:

(1) Prevention of plasma loss.—Although the use of a wave with a long wavelength provides an easier coupling and makes possible the use of a cheaper power source, it may provide a large-scale perturbation in the plasma, causing unde-

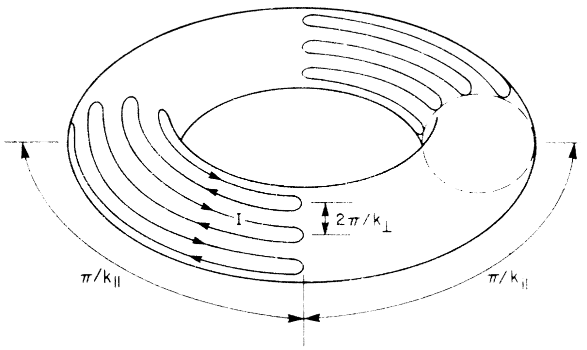

FIG. 1. Schematic diagram of the proposed setting of a heating coil using shear Alfvén wave resonance. Many turns in the poloidal direction are shown to emphasize the desirability of having $k_\perp > k_\parallel$ to cut off the magnetosonic wave. However, in the real system even if $m \sim 1$ or 2, this condition is satisfied.

sirable loss of the plasma. The use of the surface wave (not a surface eigenmode) rather than the magnetosonic wave will substantially reduce this problem.

(2) Uniform heating.—To prevent a localized heat deposit, the frequency must be swept so that the resonant condition $\omega_0 = k_\parallel v_A(x)$ is satisfied for major portions of the plasma.

(3) Coupler design.—Because one cannot place a metal boundary inside the plasma, the standing wave in the parallel direction, which is essentially needed for the resonant condition, must be provided by the coupler design. One way is to provide a periodic coil in the toroidal direction wound parallel to the toroidal axis as shown in Fig. 1. One must chose k_y smaller than $1/\rho_i$ to prevent finite Larmor-radius coupling between shear and compressional waves.

(4) Coupling to electrons.—The present method uses a one-fluid approximation of the plasma. Hence the dissipated energy goes presumably to ions. However, if a charge separation is produced parallel to $\vec{B}_0$, electrons are involved and may be heated. This may be avoided if the wave frequency is chosen to be higher than the electron drift wave frequency, $(k_\perp v_{the}{}^2/\omega_{ce}\, d\ln\rho_m/dx$.

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[4] The surface wave here means a cutoff electromagnetic wave whose wave equation is given by $\nabla^2\varphi = 0$ (φ is the magnetic potential), and *not* a surface eigenmode of the plasma.

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[8] We have assumed $k_\perp > k_\parallel$ so that the magnetosonic wave is cut off. However, because $k_\parallel \sim$ (major radius)$^{-1}$ for a toroidal system, while $k_\perp \sim$ (minor radius)$^{-1}$, even for $m = 1$, our condition is satisfied. Hence the field intensity b_{s0} is large and an efficient coupling is expected.

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