[2]H. Risken, Z. Phys. 251, 231 (1972).

[3]For economy of expression we refer to the initial value as the zeroth derivative.

[4]A sequence of numbers μ_n form a Stieljes sequence if they can be written as the moments of some probability density $\rho(x)$, that is, $\mu_n = \int_0^\infty x^n \rho(x)\,dx$. For Fokker-Planck systems, $\rho(x)$ can be expressed in terms of the eigenvalues h_n and eigenfunctions Y_n of H as $\rho(x) = \sum_n |\langle\langle Y_0 f Y_n \rangle\rangle|^2 \delta(x-h_n)$.

[5]R. G. Gordon, J. Math. Phys. (N.Y.) 9, 655 (1968), gives a very readable exposition of these points and numerous citations to the literature; *The Padé Approximant in Theoretical Physics,* edited by G. A. Baker and J. L. Gammel (Academic, New York, 1970); J. A. Shohat and J. D. Tamarkin, *The Problem of Moments,* Mathematical Surveys, No. 1 (American Mathematical Society, Providence, R. I., 1950). In the usual discussions the mathematical quantity considered corresponds to the Laplace transform of the correlation function. This Laplace transform admits an expansion as a continued fraction whose approximants are certain rational functions. Peeling off the Laplace transformation from these arguments yields the assertions made here.

[6]This is the procedure given an even number of initial derivatives; for an odd number the procedure is similar except that $c_0 = 0$.

[7]R. D. Hempstead and M. Lax, Phys. Rev. 161, 350 (1967); M. Lax and M. Zwanziger, Phys. Rev. A 7, 750 (1973); H. Risken and H. D. Vollmer, Z. Phys. 201, 323 (1967); H. Risken, Fortschr, Phys. 16, 261 (1968); C. D. Cantrell and W. A. Smith, unpublished.

[8]E. Jakeman and E. R. Pike, J. Phys. A: Proc. Phys. Soc., London 4, L56 (1971), have called attention to this fact.

Spherical Solitons*

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We derive a modified Korteweg–de Vries equation appropriate to small-amplitude, spherically symmetric waves. A numerical solution is obtained which differs qualitatively from the one-dimensional soliton solution.

Soliton solutions[1] in one dimension are now well known for acoustic waves propagating in a collisionless plasma of warm electrons and cold ions.[2] Experimental observation of this phenomenon is well founded.[3] The two-component (electrons and ions) fluid equations, together with Poisson's equation, reduce to the Korteweg–de Vries equation[4] in the small-amplitude approximation. The solution is a symmetric pulse moving with constant velocity, for which the square root of the peak amplitude multiplied by the width takes on a characteristic value.

In this Letter, we report results recently obtained by working with the three-dimensional, spherically symmetric version of this problem. We follow the procedure used in Ref. 2. The system of equations describing the motion is

$$\partial n/\partial r = -(e/kT)En, \tag{1}$$

$$\partial V/\partial t + V\,\partial V/\partial r = (Ze/M)E, \tag{2}$$

$$r^{-2}(\partial/\partial r)(r^2 E) = 4\pi e(ZN - n), \tag{3}$$

$$\partial N/\partial t + r^{-2}(\partial/\partial r)(r^2 N V) = 0. \tag{4}$$

N is the ion density, Ze the ion charge, M the ion mass, n the electron density, T the electron temperature, E the electric field (radial), V the ion fluid velocity, r the radial distance, and t the time. A stationary, isothermal electron fluid has been assumed.

We investigate ingoing solutions of Eqs. (1)–(4) in the small-amplitude approximation. The dispersion relation for acoustic waves in the linear approximation for long wavelengths leads us to define new dimensionless coordinates

$$\xi = -\sqrt{\epsilon}\,(r/\lambda_D + \omega_i t), \tag{5}$$

$$\eta = \epsilon^{3/2}\omega_i t, \tag{6}$$

where ϵ is the expansion parameter, λ_D the Debye length, and ω_i the ion plasma frequency. We transform Eqs. (1)–(4) from the coordinates (r,t) to the (ξ,η). Then we expand in powers of ϵ:

$$n = n_0 + n', \tag{7}$$

$$N = (1/Z)(n_0 + N'), \tag{8}$$

$$n' = \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \dots, \tag{9}$$

$$N' = \epsilon N^{(1)} + \epsilon^2 N^{(2)} + \dots, \tag{10}$$

$$V = \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \dots, \tag{11}$$

$$\tilde{E} = \epsilon \tilde{E}^{(1)} + \epsilon^2 \tilde{E}^{(2)} + \dots, \tag{12}$$

where

$$E = \sqrt{\epsilon}\tilde{E}. \tag{13}$$

We solve for a propagating disturbance $(n', N', V, \tilde{E})$ in a stationary background, with uniform electron density n_0. The first-order equations give

$$n^{(1)} = N^{(1)}, \tag{14}$$

$$\tilde{E}^{(1)} = [(4\pi n_0 kT)^{1/2}/c_s]\partial V^{(1)}/\partial \xi, \tag{15}$$

and

$$n^{(1)} = -(n_0/c_s)V^{(1)}, \tag{16}$$

where $c_s = (ZkT/M)^{1/2}$ is the sound speed. Actually, an arbitrary function of η can be added to the right-hand side of Eq. (16). The effect of such a term has been investigated in the literature[5] and does not change the essential properties of the solution.

The second-order equations, together with Eqs. (14)-(16), yield the following equation for the dimensionless first-order ion fluid velocity:

$$\frac{\partial U}{\partial \eta} + \frac{U}{\eta} + U\frac{\partial U}{\partial \xi} + \frac{1}{2}\frac{\partial^3 U}{\partial \xi^3} = 0, \tag{17}$$

where

$$U = -V^{(1)}/c_s. \tag{18}$$

Thus we obtain a Korteweg-de Vries equation plus an additional term U/η.

A solution was obtained by numerically integrating Eq. (17) using an initial condition corresponding to a one-dimensional soliton,

$$U(\eta_0, \xi) = 3\,\mathrm{sech}^2(\xi/\sqrt{2}). \tag{19}$$

Without the U/η term in Eq. (17), the initial pulse would propagate to larger ξ values without changing its amplitude or shape, with a propagation velocity 1.0. With the U/η term present, our spherical soliton develops according to Figs. 1(a)-1(c). As time increases and the solution propagates to larger ξ, the value of U increases, the width decreases, and a small stationary residue is left behind the advancing pulse. In the lab frame, the pulse moves inward at an increasing speed somewhat greater than sound speed. At the same time, a small residue develops and moves inward behind the pulse at sound speed. It is found that the square root of the peak amplitude multiplied by the width (full width at half-maximum) of the pulse is constant to within 2% over the entire run $-31.6 \leqslant \eta \leqslant -6.6$.

The numerical solution of Eq. (17) is based on a two-level finite difference approximation method. The scheme is similar to the one used by Zabusky and Kruskal.[1] The difference approximation for the third derivative limits the size of $\Delta\eta$ in the integration, such that the condition $\Delta\eta \leqslant 0.7698(\Delta\xi)^3$ must be satisfied for numerical stability. This limit is obtained from a stability analysis and is observed in computational tests. Periodic boundary conditions in ξ were used.

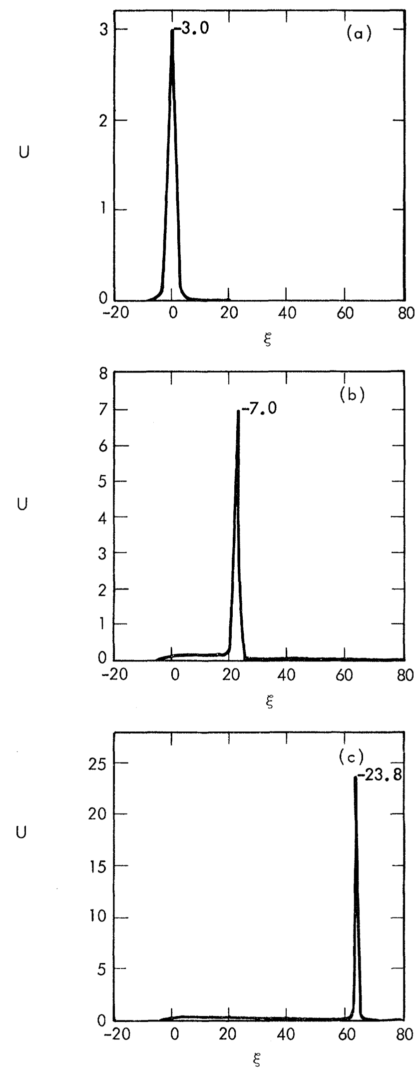

FIG. 1. Development of spherical soliton. Dimensionless U versus spatial coordinate ξ at times (a) $\eta = -31.6$ (initial condition), (b) $\eta = -16.6$, and (c) $\eta = -6.6$.

The accuracy of the numerical results was tested in several ways. Zabusky[6] obtained an exact analytical solution for a special case in which the U/η term is not present. A run with the U/η term removed from the program and $\Delta\xi = 0.1$ reproduced this analytical solution with an error of less than 1%. With the U/η term present, we compared runs with $\Delta\xi=0.2$, 0.1, and 0.08. The observed agreement was $\lesssim 1\%$ in the position of the peak and $\lesssim 0.3\%$ in the value of the peak velocity. For the run with $\Delta\xi=0.08$, the momentum and energy were conserved to within 2×10^{-6}.

An approximate solution to Eq. (17) for early times is

$$U^{(1)}(\eta,\xi)=U_0(\eta_0/\eta)\,\mathrm{sech}^2\{(U_0\eta_0/6\eta)^{1/2}[\xi-\xi_0-\tfrac{1}{3}\eta_0U_0\ln(\eta/\eta_0)]\}, \tag{20}$$

where $U_0=U(\eta_0,\xi_0)$. According to Eq. (20), the peak should move so that

$$\xi^{\mathrm{pk}}(\eta)=\xi^{\mathrm{pk}}(\eta_0)+\tfrac{1}{3}\eta_0U_0\ln(\eta/\eta_0). \tag{21}$$

Indeed, this relationship holds for $-31.6\leq\eta\lesssim-26.6$. In order to describe correctly the motion of the peak, an additional term $\frac{1}{2}\eta_0U_0(1/\eta-1/\eta_0)$ was needed for $-26.6\leq\eta\lesssim-11.6$. In the range $-31.6\leq\eta\lesssim-26.6$, allowing for a small displacement in ξ, the solution $U^{(1)}$ from Eq. (20) agrees with the numerical solution to ~5%.

We can find the evolution in time of any point in the propagating pulse from a knowledge of the characteristic of that point in the (ξ,η) plane. We solve the equation

$$\partial U^{(c)}/\partial\eta+U^{(c)}/\eta+U^{(c)}\partial U^{(c)}/\partial\xi=0 \tag{22}$$

to obtain

$$\eta U^{(c)}(\eta)=\frac{\eta_0U_0^{(c)}+A^{(c)}[\xi^{(c)}(\eta)-\xi^{(c)}(\eta_0)]}{1+A^{(c)}\ln(\eta/\eta_0)}. \tag{23}$$

If $U^{(c)}(\eta)=U^{(\mathrm{pk})}(\eta)$, $\xi^{(\mathrm{pk})}(\eta)$ is the position of the peak at time η. The constant of integration, $A^{(c)}$, is different for different points $U^{(c)}$. The formula (23) has been checked for several U values over the entire range $-31.6\leq\eta\leq-6.6$ and works quite well.

It is worthwhile to remark first that the early-time solution [Eq. (20)] predicts a constant value for the square root of the peak amplitude multiplied by the width, and this is observed in the numerical solution for all times. Secondly, a comparable amount of momentum is contained in the residue as in the pulse at the end of the run ($\eta=-6.6$). However, only a negligible portion of the energy of the initial pulse is "shared" with the residue. Finally, the main part of the solution, contained in the pulse, grows faster than η^{-1} and propagates faster than a corresponding one-dimensional soliton with the same amplitude.

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Quasielastic Rayleigh Scattering in a Smectic-*A* Crystal*

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Using a light-beating spectrometer, we have observed in the smectic-*A* phase of *p*-cyanobenzilidene-*p*-octyloxyaniline the thermally excited undulations of layers predicted by de Gennes. From the wave-vector dependence of their damping time we determined the diffusivity of the angular orientation, $K_1/\eta=(2.0\pm0.2)\times10^{-6}$ cgs. From the thickness dependence of the boundary quenching, we determined the penetration length, $\lambda=14\pm1$ Å at 75°C.

Smectic-*A* liquid crystals are systems of liquid layers which can be aligned parallel to plane glass boundaries. As shown by de Gennes[1] these layers should undulate easily under thermal excitation of wave vector $\vec{q}$ parallel to the layers. These thermally excited undulations should give rise to a strong depolarized quasielastic Rayleigh scattering. Intense but purely elastic light