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[13]We adopt here the standard definitions of homogeneous and inhomogeneous broadening. For homogeneous broadening the lifetime of the molecular state is limited by the broadening mechanism. Radiative broadening is of course homogeneous, and *inelastic* collisions, in which the quantum state of the molecule is changed, also serve to broaden the line homogeneously. On the other hand an inhomogeneously broadened line is one in which the center frequency of the oscillators is distributed over a line profile, and the lifetime is the same throughout. Doppler broadening is inhomogeneous and *elastic* collisions, in which the phase of the oscillator is interrupted but the quantum state is unchanged, also result in inhomogeneous broadening.

# Dielectric Function and Diffusion of a Guiding-Center Plasma*

J. B. Taylor†
*Institute for Advanced Study, Princeton, New Jersey 08540*
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The dielectric function for the two-dimensional guiding-center plasma is discussed. An error is noted in a calculation based on the Vlasov equation by Lee and Liu and an improved derivation is given using the Kubo formalism. Although this new dielectric function embodies the correct screening effect, it does not (contrary to earlier suggestions) essentially change the Taylor-McNamara result for the diffusion coefficient.

The two-dimensional guiding-center (2D gc) plasma[1] (in which charged filaments are aligned with a magnetic field $\vec{B}$ and move with the velocity $\vec{E}\times\vec{B}/B^2$) is important for the study of plasma diffusion due to thermal vortices.[1-6] In such calculations the dielectric function is often invoked.

In a conventional plasma the dielectric function is obtained from the linearized Vlasov equation by integrating the effect of the perturbing potential $\nabla\varphi\cdot(\partial f^0/\partial\vec{v})$ along noninteracting, unperturbed particle orbits.[7] This yields the plasma response $\tilde{\rho}(\vec{x},t)$, and the susceptibility $\chi^V(\vec{k},\omega)$ is defined by

$$4\pi\tilde{\rho}(\vec{k},\omega) = -k^2\chi^V(\vec{k},\omega)\varphi(\vec{k},\omega).$$

Then the Vlasov-based dielectric function is

$$\epsilon^V(\vec{k},\omega) = 1 + \chi^V(\vec{k},\omega).$$

For the 2D gc plasma the analog of the Vlasov equation is

$$\frac{\partial F}{\partial t} + \frac{\vec{E}\times\vec{B}}{B^2}\cdot\frac{\partial F}{\partial\vec{X}} = 0,$$

and the dielectric function would be obtained by integrating

$$(\vec{B}\times\nabla\varphi/B^2)\cdot\partial F_0/\partial\vec{X}$$

along the noninteracting unperturbed particle orbits. However, these orbits are of zero length since in the absence of interaction each particle remains at rest. Furthermore, for a uniform plasma $\partial F_0/\partial\vec{X} = 0$, $F_0^+ = F_0^-$, and the perturbation can produce no charge imbalance. Hence, for the gc plasma the Vlasov-like approximation yields only $\chi^V = 0$ and $\epsilon^V = 1$.

In an attempt to obtain an improved value for $\epsilon$, Lee and Liu[4] considered a plasma with a finite gyroradius $a$ and invoked the now well-known[8-11] device of introducing diffuse orbits in place of the precise free orbits. This is intended to simulate the effect of particle interactions. On taking the limit $a \to 0$ one then finds for the dielectric function of the gc plasma

$$\epsilon(\vec{k}, \omega) = 1 + (k_D^2/k^2)[1 - i\omega/(i\omega - k^2 D)], \tag{1}$$

where the diffusion coefficient $D$ arises from the smeared-out orbits, which introduce a factor $\sim \exp(-k^2 D\tau)$ into the orbit integration, and $k_D^{-1}$ is the Debye length. Because of the screening effect embodied in (1) a calculation using this dielectric function leads to a finite value for the diffusion coefficient,[4] unlike the original calculation of Taylor and McNamara[1] in which $D$ is logarithmically divergent at long wavelength.

However, it is clear that the introduction of diffuse orbits into the calculation cannot *of itself* generate a nonzero $\chi^V$ for the gc plasma, since one still has $\partial F_0/\partial \vec{X} = 0$ and $F_0^+ = F_0^-$. In fact it appears that the nonzero value found by Lee and Liu[4] is a consequence of treating the unperturbed distribution function as constant during the orbit integration. This is inconsistent with the introduction of diffuse orbits. [Dawson and Okuda[12] have also pointed out that according to (1) the damping of normal modes, defined by $\epsilon(\vec{k},\omega) = 0$, would be $\gamma = (k^2 + k_D^2)D$, whereas their numerical simulations show that $\gamma \propto k^2$ even for $k^2 \ll k_D^2$.]

Clearly, the Vlasov equation is not a suitable starting point for the calculation of $\epsilon(\vec{k}, \omega)$ for a gc plasma. However, there is a more accurate procedure, due to Kubo,[13] into which the concept of orbit diffusion can be introduced naturally and consistently. The dielectric function thus calculated differs from (1) and yields a damping rate $\propto k^2$. It also yields the correct thermal spectrum $\langle E_k^2 \rangle$ and the correct plasma screening (both of which are known independently from thermodynamic considerations), yet it leaves the diffusion coefficient essentially unchanged from the Taylor-McNamara value.

Although unconventional, the gc system can nevertheless be described in Hamiltonian form. The Hamiltonian function is the interaction energy $\sum e_i e_j \ln|\vec{r}_{ij}|$, and the Cartesian coordinates $x_i$ and $y_i$ are, apart from scale factors, canonically conjugate to one another. For brevity the full set of coordinates is denoted by $\{\vec{X}_i\}$. Then the many-particle distribution function $F\{\vec{X}_i\}$ obeys the Liouville equation

$$\partial F/\partial t = [H, F],$$

and the linear response to an external potential $e^{i\omega t}\varphi(x)$ is given by

$$\partial F^{(1)}/\partial t = [H_0, F^{(1)}] + e^{i\omega t}[\Phi, F^0], \tag{2}$$

where $H_0$ is the Hamiltonian of all the interparticle interactions and $\Phi\{\vec{X}_i\}$ is the sum of all interactions with the *external* potential. The function $F^0\{\vec{X}_i\}$ is the thermal-equilibrium distribution $\exp(-\beta H_0)$, so that Eq. (2) becomes

$$\partial F^{(1)}/\partial t - [H_0, F^{(1)}] = \beta F_0 [H_0, \Phi] e^{i\omega t}.$$

Recognizing that the left-hand side of this equation is the derivative along *interacting* orbits and that $F_0$ is constant along these orbits,

$$F^{(1)}\{\vec{X}_i\} = -\beta F_0 \Phi e^{i\omega t} + i\omega\beta F_0 \int^t e^{i\omega\tau}\Phi(\tau)\, d\tau,$$

where $\Phi(\tau) \equiv \Phi\{\vec{X}_i(\tau)\}$ and $\{\vec{X}_i(\tau)\}$ denotes the positions at time $\tau$ of interacting particles which arrive at $\{\vec{X}_i\}$ at time $t$.

The average change in any function $B\{\vec{X}_i\}$ caused by the external potential is

$$\langle B \rangle = \int B\{\vec{X}_i\} F^{(1)}\{\vec{X}_i\} \prod d^2 X_i,$$

so that

$$\langle B \rangle = e^{i\omega t}[-\beta\langle B\Phi\rangle_0 + i\omega\beta \int_{-\infty}^0 e^{i\omega s}\langle B(0)\Phi(s)\rangle_0\, ds],$$

where $\langle\ \rangle_0$ denotes an average over the *thermal* distribution:

$$\langle Z \rangle_0 \equiv \int \exp(-\beta H_0\{\vec{X}_i\}) Z\{\vec{X}_i\} \prod d^2 X_i.$$

To find the dielectric constant it is now only necessary to take

$$\Phi\{\vec{X}_i\} = \hat{\varphi}_k \sum_i e_i \exp[-i\vec{k}\cdot\vec{X}_i(t)] = \hat{\varphi}_k \rho_{-k}(t),$$

representing a single Fourier mode, and to determine the resulting induced charge by taking

$$B\{\vec{X}_i\} = \tilde{\rho}_{\vec{k}} = \sum_i e_i \exp(+i\vec{k}\cdot\vec{X}_i).$$

Then

$$\tilde{\rho}_{\vec{k}}(\omega) = \hat{\varphi}_k(\omega)[-\beta\langle\rho_{-k}(0)\rho_k(0)\rangle_0 + i\omega\beta\int_{-\infty}^0 e^{i\omega s}\langle\rho_{-k}(0)\rho_k(s)\rangle_0\, ds]. \tag{3}$$

It is now important to recall that $\hat{\varphi}$ is the *external* potential and not, as in the Vlasov calculation, the total potential. Hence, if we define an *external* susceptibility by

$$4\pi\tilde{\rho}(\vec{k}, \omega) = -k^2\chi^T(\vec{k}, \omega)\hat{\varphi}(\vec{k}, \omega), \tag{4}$$

the dielectric function is

$$\epsilon^{T}(k,\omega)=[1-\chi^{T}(\vec{k},\omega)]^{-1}, \tag{5}$$

with $\chi^{T}$ given by

$$\chi^{T}(\vec{k},\omega)=(4\pi/k^2)[\beta\langle|\rho_k{}^2|\rangle_0-i\omega\beta\int_{-\infty}^{0}e^{i\omega s}\langle\rho_{-k}(0)\rho_k(s)\rangle_0\,ds]. \tag{6}$$

The distinctive feature of (6) is that the integration is along interacting orbits whereas $\chi^{V}$ was obtained by integrating along free orbits. If one were simply to neglect interactions, then it is true that $\chi^{T}\to\chi^{V}$, but one does *not* thereby recover the Vlasov dielectric function since $\epsilon^{T}=1/(1-\chi^{T})$ but $\epsilon^{V}=(1+\chi^{V})$. The correct method of passing from the Kubo form to the Vlasov limit has been discussed by Balescu[14] using diagrammatic techniques.

The thermal spectrum $\langle|\rho_k{}^2|\rangle_0$ is given by $ne^2k^2/(k^2+k_D{}^2)$ and for the gc plasma the correlation function

$$\langle\rho_{-k}(0)\rho_k(t)\rangle_0=\sum_{ij}\langle e_ie_j\exp\{-i\vec{k}\cdot[\vec{X}_i(0)-\vec{X}_j(t)]\}\rangle_0$$

is determined by the fluctuating electric field (through the velocity $\vec{E}\times\vec{B}/B^2$). In the model of Taylor and McNamara it is assumed that these fluctuations are normally distributed. Then

$$\langle\rho_{-k}(0)\rho_k(t)\rangle_0=\langle|\rho_k|^2\rangle_0\exp[-k^2R(t)], \tag{7}$$

where $R(t)$ is the rms particle displacement in time $t$. This has been calculated numerically[1] but is approximately $R=Dt$. It is clear that the correlation function must be of the form (7) whenever the dispersion of particles can be described as diffusion.

Using this approximation the dielectric function for the gc plasma is given by

$$\frac{1}{\epsilon(\vec{k},\omega)}=\left(1-\frac{k_D{}^2}{k^2+k_D{}^2}\frac{k^2D}{i\omega+k^2D}\right). \tag{8}$$

This has several noteworthy features. Firstly, the damping of normal modes is $\gamma_{\vec{k}}=k^2D$. Secondly, the static screening factor $\epsilon^{-1}(\vec{k},0)$ is given correctly as $\epsilon^{-1}(\vec{k},0)=k^2/(k^2+k_D{}^2)$. Thirdly, the derivation of (8) ensures that, in conjunction with the fluctuation-dissipation theorem

$$\langle|E^2(\vec{k},\omega)|\rangle/8\pi=(\kappa T/2\pi)\,\mathrm{Im}[\omega\epsilon(\vec{k},\omega)]^{-1}, \tag{9}$$

it will yield the correct thermal spectrum, i.e.,

$$\langle|E_k{}^2|\rangle/8\pi=\tfrac{1}{2}\kappa Tk_D{}^2/(k^2+k_D{}^2). \tag{10}$$

We must now consider the effect of the dielectric function (8) on the calculation of the test-particle diffusion coefficient. This can be written[4]

$$D=(c^2/B^2)\sum_{\vec{k},\omega}\langle|E^2(\vec{k},\omega)|\rangle\int_0^\infty d\tau\langle\exp(i\omega\tau+i\vec{k}\cdot\vec{x}(\tau))\rangle, \tag{11}$$

where it has been assumed, as usual, that averages over the plasma ensemble and over the test particle are taken separately. Treating the test-particle dispersion in the diffusion approximation, it is found that

$$D=\frac{c^2}{B^2}\sum_{\vec{k},\omega}\langle|E^2(k,\omega)|\rangle\frac{k^2D}{\omega^2+(k^2D)^2}. \tag{12}$$

The frequency-dependent fluctuation spectrum $\langle|E(\vec{k},\omega)|^2\rangle$ can be found from the dielectric function by the fluctuation-dissipation theorem (9); then

$$D=\frac{c^2}{B^2}\frac{\kappa T}{\pi^2}\int d^3k\,d\omega\frac{k_D{}^2}{k^2+k_D{}^2}\left(\frac{k^2D}{\omega^2+k^4D^2}\right)^2, \tag{13}$$

or

$$D^2=\frac{c^2}{2B^2\pi}\kappa T\int\frac{d^3k}{k^2}\frac{k_D{}^2}{k^2+k_D{}^2}. \tag{14}$$

This result retains the long-range logarthmic divergence of the original Taylor-McNamara result, *despite* the fact that the dielectric function embodies the plasma screening effect. In fact, the diffusion coefficient given by (14) differs by only a factor $\sqrt{2}$ from the original result.[1] A similar modification was noted by Montgomery[15] and has been incorporated in some calculations.[5]

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†Permanent address: Culham Laboratory, Abingdon, Berkshire, England.

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## Pressure-Balance Limitations in $Z$ Pinches with Diffusion Heating*

D. A. Baker and J. A. Phillips
*Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544*

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Conditions of global energy and pressure balance are applied to a cylindrically symmetric plasma column having a $B_\theta$ confining field and a $B_z$ stabilizing field, as exist in stabilized $z$ pinches and tokamaks. In the process of producing stable configurations the plasma pressure due to Ohmic heating during magnetic field diffusion may become too large for the confining field to contain.

In fusion research there is revived interest in $z$ pinches having magnetic field configurations with diffuse profiles since stable configurations exist which confine high-$\beta$ plasmas according to magnetohydrodynamic (MHD) theory.[1-3] Experimentally the poloidal field diffuses rapidly and most of the profiles now being considered have longitudinal currents nearly uniform inside the pinched-plasma column. In many experiments the plasma $\beta_\theta$'s are high. The production of stable configurations would be greatly simplified by operating at lower values which by MHD theory would have a greater margin of stability. Attempts, however, to lower the plasma energy density have not been completely successful.

These considerations suggest that there are limitations on $\beta_\theta$ due to the processes involved in setting up the plasma and field profiles. In particular, the plasma is Ohmically heated during the diffusion of the poloidal field. For an incompressible, constant-conductivity, plane conductor excited by a step-function current the result[4] that the local Joule heating is greater than the field energy density led early workers on the diffuse pinch to the prediction[5] that $\beta_\theta > \frac{1}{3}$ without losses. These considerations led to the expectation that unless special measures were taken the Joule heating would raise the $\beta$ above the low values then believed necessary for stability. A more detailed cylindrical calculation[6] on the incompressible model gives heating corresponding to $\beta_\theta > 2$, at the time the field has diffused so that it is uniform within 10% across the conductor, for all values of the current rise time. This result indicates the possibility that in the formation of a $z$-pinch plasma, heating by diffusion prevents magnetic containment by exceeding pressure-balance conditions as well as those for stability. This paper treats this problem using general energy- and pressure-balance conditions