action between Maxwell molecules for which $\lambda = v/\nu$, then for any field strength, the distribution is always Maxwellian,

$$f_0 = A \exp\left[- \frac{mv^2}{2kT_n' + M\gamma^2/3(\omega^2+\nu^2)} \right],$$

with a modified "temperature" at

$$T_n' = T_n + \frac{MU^2}{3k} + \frac{M\gamma^2}{6k(\omega^2+\nu^2)}.$$

In this case the energy spread is determined not only by the neutral-gas temperature and the flow speed, but also by the field strength, the frequency, and the collision rate.

(II) For arbitrary angle α, the results are similar for subcases (A) and (B), except that v^2 is replaced by $(3U^2\cos^2\alpha + v^2)$. Effectively, the electron-distribution function is now further broadened by the component of the flow velocity in the field direction. For the corresponding subcase (C) of Case (I), Eq. (14) can also be integrated exactly.

Once f_0 is obtained, we can apply Eq. (8) to calculate the conducitivity, current density, dielectric constant, and index of refraction in a weakly ionized flowing gas. Detailed application to the microwave-plasma interaction will be given in another paper.

The author would like to thank Professor B. S. Tanenbaum for suggesting this problem to him and for useful comments. He is also grateful to Dr. S. Humphries, Jr., for helpful discussions.

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Electron and Ion Heating Through Resonant Plasma Oscillations

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(Received 8 April 1974)

We present an analytic solution for the resonant oscillations of a cold, inhomogeneous, one-dimensional plasma under the influence of a sinusoidal applied field. The model approximates the response to obliquely incident electromagnetic radiation. Energy can be absorbed collisionlessly through either electron wave breaking or ion bunching, when the driven field is strong or weak, respectively. The two regimes are distinguished by $eE_L \gtrless 0.02Lm^2\omega^2/M$, where L is the plasma scale length and E_L the longitudinal driver. In the weak-driver case, ions with keV energies can be produced in the resonance region.

Particle-code simulations[1] show that the resonant response of a one-dimensional plasma to obliquely incident electromagnetic radiation[1-6] is well approximated by its response to a longitudinal field applied parallel to its density gradient. The computer results, and the analytic solution presented here, predict large-amplitude electrostatic waves propagating in the direction of decreasing density; such waves have recently been observed experimentally.[2]

The present analysis is based on cold-plasma equations and small ion-density perturbations; the dominant mechanism is that of driven-electron–plasma oscillations which exhibit wavelike behavior—phase and amplitude variations caused by the inhomogeneous background ion density. They are not Langmuir waves capable of convecting energy away from the resonant region.[2-5] The theory is valid if the nonresonant-electron "quiver" velocity is greater than the thermal speed, $eE_L/m\omega > v_{te}$, where E_L is the *longitudinal* driver field; computer studies[6] indicate that this is a conservative criterion.

Solutions based on this model fail when the

electron waves break or when ion-density perturbations become nonnegligible. Electron wave breaking generates a stream of fast electrons,[7] which invalidates the cold-plasma assumption; physically this means the absorption of incident energy through collisionless electron heating. On the other hand, ion bunching spoils the electron resonance; it may also produce ion overtaking and thus fast ions.[7] Therefore, the two ways in which the theory breaks down correspond to regimes of preferential electron and ion heating.

The driven-electron oscillations behave at first as if the ions were immobile, i.e., the local initial ion density $N_0(x)$ causes simple harmonic motion of the electrons with frequency $\omega_p(x)$ (provided the amplitude is small enough for the oscillators to remain linear; analysis shows that this is true). The oscillation amplitude grows secularly in a small, shrinking region around the critical surface, bounded by two principal nodes.

In these cold-electron-plasma oscillations, the electric field is proportional to the displacement, $E(x,t) \propto \delta(x_0,t)$, where x_0 is the equilibrium position (the Lagrangian coordinate[8]) of the electron presently at x. Because the amplitude peaks on resonance, electrons coming from the direction of the critical surface "carry with them" larger electric fields than those coming from the opposite direction. The resulting time-average electric field pushes the ions away from the critical surface.

Ions leaving the neighborhood of the critical surface are first accelerated, then decelerated (the force described above reverses itself beyond the principal nodes, where the amplitude of the oscillations increases again). In traversing the region of decreasing acceleration, they tend to bunch,[9] and positive ion-density perturbations occur near the boundaries of the resonant region. These, and the negative perturbation at the critical surface, grow with time, and eventually can no longer be ignored.

There are two time scales of interest: t_b, the time at which the electron waves break; and t_i, the time when ion-density perturbations must be accounted for. If $t_b < t_i$, energy is put into fast electrons, while if $t_i < t_b$, the ion perturbations cause an end to the resonant transfer of energy into the oscillations. By the time this happens, the heavy ions have acquired a drift velocity which causes them to overtake unless the fields reverse themselves sufficiently.

Steady-state analysis of resonant absorption[10] predicts a ponderomotive force that pushes the ions away from the critical surface. However, in the collisionless case the steady-state field is infinite at the critical surface.[3-5] We show here that wave breaking and ion bunching preclude the existence of such a steady-state solution.

Analysis of the electron motion is straightforward in Lagrangian coordinates.[8] The equation of charge continuity and Poisson's equation in one dimension admit a homogeneous driver field, which we take to be $E_L \sin\omega t$. These two equations and the momentum equation for cold electrons are sufficient to determine the electron displacement $\delta = x - x_0$, if we assume that the ion-density perturbation ΔN is small compared with N_0. If distances are measured in units of L and times in units of ω^{-1}, the equation for δ is

$$\ddot{\delta} + \omega_p^2(x_0)\delta = \epsilon \sin t, \tag{1}$$

where $\omega_p^2 = \pi e^2 N_0/m\omega^2$, and $\epsilon = eE_L/m\omega^2 L$ is a small parameter. Equation (1) is valid if $\delta \ll 1$. Its solution is conveniently written as

$$\delta = \frac{\epsilon}{(2+\nu)(1+\nu)}\left[\sin t - \frac{\sin\frac{1}{2}\nu t}{\frac{1}{2}\nu}\cos(1+\tfrac{1}{2}\nu)t\right], \tag{2}$$

where $\nu = \omega_p - 1$. The limit of δ as $\nu \to 0$ (at the resonance) is

$$\delta = \tfrac{1}{2}\epsilon(\sin t - t\cos t), \tag{3}$$

displaying a secular growth. The second term in Eq. (2), which corresponds to the secular term in Eq. (3), shows that the oscillations also grow in a contracting region about the critical point defined by two principal nodes located at $\pm\nu_n$, where

$$\nu_n = 2\pi/t. \tag{4}$$

The phase of the resonant term in Eq. (2) is such that the displacement at positive ν leads the displacement at the critical surface, while that at negative ν lags it. Therefore, in Lagrangian space, the displacement resembles a wave propagating from supercritical to subcritical ν, traversing the internodal distance in one driver period.

After the transformation from Lagrangian to Eulerian coordinates is carried out, through inversion of

$$x(x_0,t) = x_0 + \delta(x_0,t), \tag{5}$$

the electric field, electron velocity, and number

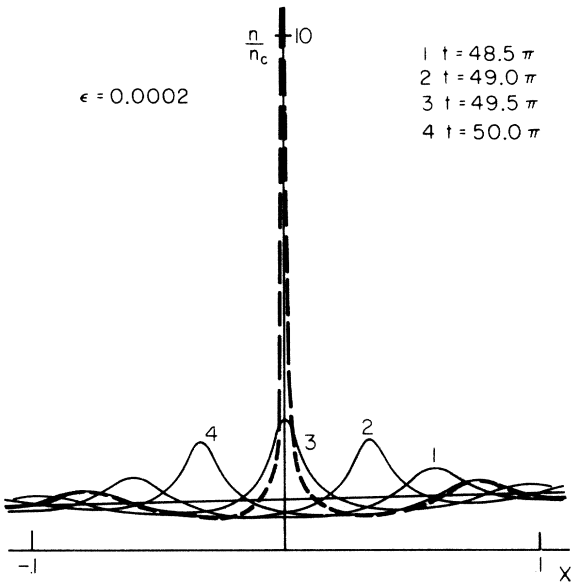

FIG. 1. The electron number density as a function of x at four different times during a driven cycle. The dashed curve is at $t = 61.5\pi$.

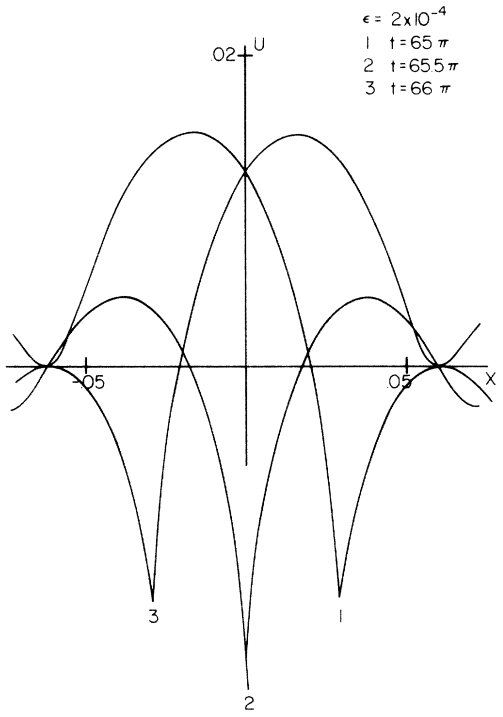

FIG. 2. The electron phase space during the cycle in which the waves break. Velocities are in units of ωL, and distance in units of L.

density are found, respectively, from

$$E(x,t) = 4\pi e L N_0(x_0)\delta(x_0,t), \tag{6}$$

$$u(x,t) = \omega L \dot{\delta}(x_0,t), \tag{7}$$

$$n(x,t) = N_0(x_0)(\partial x/\partial x_0)^{-1}. \tag{8}$$

In Eq. (6) the driver term, small compared with the resonant term, has been dropped.

These fields exhibit the properties of the particle-code simulation of Friedberg *et al.*[1] Waves of secularly increasing amplitude propagate from overdense to underdense regions, growing as they approach the critical surface and decaying as they recede from it (Figs. 1 and 2). They eventually break; the condition for breaking is[7] $\partial x/\partial x_0 = 0$. Differentiation of Eq. (3) shows that $\partial x/\partial x_0$ has its minimum value for each cycle at the critical surface, and that wave breaking occurs near $t = t_b$, where

$$\epsilon t_b^2 = 8. \tag{9}$$

At that time, the magnitude of δ is $(2\epsilon)^{1/2}$.

The ion equation of motion in dimensionless units is

$$\ddot{X} = (m/M)\delta(x_0,t), \tag{10}$$

where x_0 is the initial position of the electron presently at the ion position X. If the ion displacement $X - X_0$ is small compared with the nodal distance, the distinction between Eulerian and Lagrangian ion coordinates is unimportant, and x_0 can be considered as a function of X_0 rather than of X. If the electron waves are not close to breaking, the inversion of Eq. (5) in this case can be approximated by

$$\delta(x_0,t) \sim \delta(X_0,t)[(1 - \partial\delta(X_0,t)/\partial X_0]. \tag{11}$$

With this substitution, Eq. (10) can be integrated. Sinusoidal terms of frequency ω or 2ω make a negligible contribution. The low-frequency and dc terms give

$$X(X_0,t) \sim X_0 + \frac{m\epsilon^2}{4M\nu^3(2+\nu)^3}\left[\frac{t^2}{2}(\omega_p^4 + 3\omega_p^2) + \frac{t\sin\nu t}{\nu}\omega_p^2(\omega_p+1) - \frac{(1-\cos\nu t)}{\nu^2}(5\omega_p^3 + 2\omega_p^2 + \omega_p)\right], \tag{12}$$

where ν and ω_p are to be evaluated at X_0.

The ion density, $N_0(\partial X/\partial X_0)^{-1}$, can be evaluated easily, at any time, along the Lagrangian node trajectory (4). We find

$$N(X(x_n(t),t),t) \sim N_0(x_n)[1 - (4\pi)^{-4} m\epsilon^2 t^6/M]^{-1}, \tag{13}$$

where $x_n(t)$ is the present location of the node in x_0 space. Numerical integration of Eq. (10) without approximation (11) shows that this is a good estimate of the maximum ion density at time t.

It is not clear within the present framework how large a density perturbation can exist without destroying the resonance, but Eq. (13) is not sensitive to this. Letting α be the tolerable $\Delta N/N_0$, assumed small, we find the time t_i at which the resonance is destroyed to be

$$t_i = (4\pi)^{2/3}(M/m)^{1/6}\epsilon^{-1/3}\alpha^{1/6}. \quad (14)$$

In order for criterion (14) to be valid, t_i must be small compared with t_b, since otherwise Eq. (11) is not true. At any rate, $t_i = t_b$ is the condition separating preferential ion and electron heating. If in Eq. (14) we take $\alpha = 1$, i.e., take t_i to be the time of ion overtaking predicted by Eq. (13), and use Eq. (9), we find a condition on the driver field

$$\epsilon = 0.02m/M. \quad (15)$$

For stronger drivers the electrons are heated while for weaker drivers the ion dynamics dominates.

Since the approximations used in deriving Eq. (13) lose their validity when $\Delta N/N_0$ is nonnegligible, we cannot assume that the force tending to bunch the ions continues to act as before. However, the ions acquire a drift motion tending to make them overtake, as can be seen from the behavior of the ion density in Fig. 3, under the assumption that all fields cease to act when $\alpha = 0.1$.

The velocity of the energetic ions generated by overtaking is at least the maximum ion velocity produced by the time-average field. This velocity is sensitive to the cutoff α. It can be estimated through differentiation of Eq. (12) and evaluation of the derivative at t_i, along the nodal trajectory (4). The ion kinetic temperature kT_i which corresponds to this velocity is

$$kT_i \sim \tfrac{9}{32}(4\pi\epsilon\alpha)^{4/3}(M/m)^{1/3}mc^2(L/\lambda)^2, \quad (16)$$

where λ is the free-space wavelength of the driver, $2\pi c/\omega$. For a deuterium plasma with $\epsilon = 2 \times 10^{-6}$ and $\alpha = 0.01$, $kT_i \sim 0.1(L/\lambda)^2$ eV, some 100 eV for $L/\lambda \sim 30$ and 1 keV for $L/\lambda = 100$.

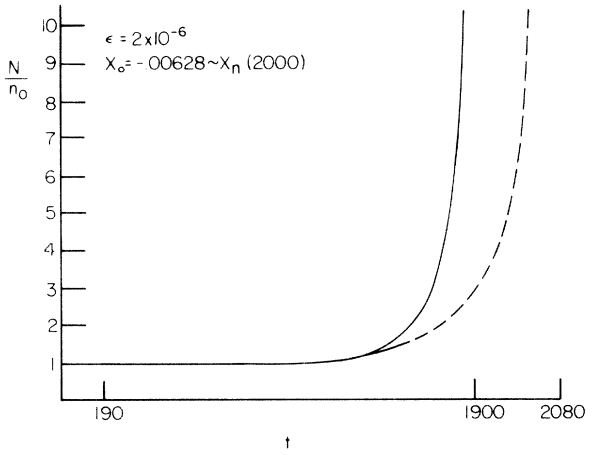

FIG. 3. The ion number density at the present position of an ion versus time, if Eq. (11) is valid until overtaking (solid curve) and if the electric field is zero after $\Delta N/N_0 = 0.1$ (dashed curve).

We gratefully acknowledge the cooperation of R. L. Morse, who provided us with motion picture output of the particle code used in Ref. 1. This work has been supported by the Laboratory for Laser Energetics, Laser Fusion Feasibility Project at the University of Rochester.

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