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*Submitted by one of the authors (V.P.W.) in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Temple University.

†Work partially supported by National Science Foundation Grant No. GT-16336.

‡National Bureau of Standards Guest from Physics Department, Temple University.

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Self-Consistent Screening Calculation of the Critical Exponent η

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(Received 1 April 1974)

A self-consistent version of the $1/n$ expansion is used to calculate the critical exponent $\eta(n,d)$ for an n-component Ginzburg-Landau field with spatial dimensionality d. The result is exact to first order in $1/n$ but also includes a partial summation of graphs to all orders in $1/n$. This leads to a bounding of η for small n, in contrast to the simple $1/n$ expansion. Results are $\eta(3,3)\simeq 0.079$, $\eta(2,3)\simeq 0.11$, and $\eta(1,3)\simeq 0.177$. For $d=2$ the theory leads to the conjecture that η vanishes for large values of n.

A recent approach[1-5] to the problem of second-order phase transitions consists of expanding the critical exponents as power series in $1/n$, where n is the number of components of the order parameter. This procedure (the "screening approximation") gives systematic corrections to the spherical model (Hartree approximation) which corresponds to the limit $n\to\infty$. At the present time exponents are known to order $1/n$ for all d in the range $2<d<4$. Unfortunately, it has thus far proved difficult to extend the expansion beyond the first order. (The exception is Abe's calculation,[2] to order n^{-2}, of η for the special case $d=3$.) It is possible, however, to include an infinite subset of such higher-order terms in a straightforward way by using self-consistently determined propagators in the graph-theoretic formulation to the problem.[6] The calculation of η within this "self-consistent screening approximation" (SCSA) is the purpose of this Letter.[7] In contrast to the simple $1/n$ expansion, the inclusion, within the SCSA, of terms of all orders in $1/n$ leads to a bounding of η for small n. In addition, the method is sufficiently powerful to deal with the case $d=2$, where the simple $1/n$ expansion breaks down (as far as the calculation of critical exponents is concerned[8]). For this case we find that η vanishes for $n\geqslant 2$. For $n<2$, a nontrivial solution appears with η increasing monotonically from zero at $n=2$ to unity at $n=0$. We conjecture that this result is qualitatively correct.

The calculation starts from the Ginzburg-Landau (GL) free-energy functional:

$$F_{\rm GL}\{\varphi\} = \int d^d r \{\tfrac{1}{2}\sum_{i=1}^{n}[\tau\varphi_i^2 + (\nabla\varphi_i)^2] + \tfrac{1}{4}n^{-1}(\sum_{i=1}^{n}\varphi_i^2)^2\}.$$

Here $\tau \propto (T - T_c)/T_c$, where T_c is the mean-field transition temperature. The order-parameter correlation function, or propagator, is given by

$$g(\vec{r}) = \langle\varphi_j(\vec{r})\varphi_j(0)\rangle = Z_{\rm GL}^{-1}\int \prod_i (d\varphi_i)\varphi_j(\vec{r})\varphi_j(0)\exp(-F_{\rm GL}\{\varphi\}),$$

where $Z_{\rm GL} = \int \prod_i (d\varphi_i)\exp(-F_{\rm GL}\{\varphi\})$. We are interested in the Fourier transform $g(k)$, given by

$$g(k) = [\tau + k^2 + \sigma(k)]^{-1}, \tag{1}$$

with $\sigma(k)$ the self-energy function. The SCSA is defined[6] by the self-energy graphs of Fig. 1(a). The straight line depicts the fully dressed propagator; the wavy line represents the "screened" potential, $-(1/n)\nu(q)$, and is given by the Dyson equation of Fig. 1(b). In the usual way each dashed line is associated with a factor $-1/n$ and each closed loop with a factor n to give[6]

$$\sigma(k) = \textstyle\sum_q g(q) + (2/n)\sum_q \nu(q) g(\vec{q}+\vec{k}) + O(1/n^2), \tag{2}$$

$$\nu(q) = [1 + \pi_0(q)]^{-1}, \tag{3}$$

where

$$\pi_0(q) = \textstyle\sum_p g(p) g(\vec{p}+\vec{q}). \tag{4}$$

To calculate the exponent η we work at the critical point, where $g(0) = \infty$. Then $g(k)$ takes the form

$$g(k) = (Z^{-1}/k^2)(k/k_c)^\eta, \tag{5}$$

with k_c and Z^{-1} constants, one of which we are free to choose. Substitution into Eq. (4) yields[9]

$$\pi_0(q) = Z^{-2} f(d,\eta) k_c^{-2\eta} q^{d+2\eta-4}, \quad 4 - 2\eta > d > 2 - \eta, \tag{6}$$

where

$$f(d,\eta) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(2-\eta-\frac{1}{2}d)[\Gamma(\frac{1}{2}d+\frac{1}{2}\eta-1)]^2}{\Gamma(d+\eta-2)[\Gamma(1-\frac{1}{2}\eta)]^2}$$

and the inequalities on η ensure that the sum for $\pi_0(q)$ converges.

For small q, $\pi_0(q) \gg 1$, so that Eq. (3) gives $\nu(q) \approx [\pi_0(q)]^{-1}$. For large q, on the other hand, $\pi_0(q) \ll 1$, giving $\nu(q) \approx 1$. The intermediate region determining the changeover between these two regimes is given by $q \sim k_c$, where $\pi_0(k_c) = 1$. This fixes k_c. Hence we adopt the following form for $\nu(q)$:

$$\nu(q) \simeq \begin{cases} Z^2[f(d,\eta)]^{-1} k_c^{2\eta} q^{4-d-2\eta}, & q < k_c, \\ 1, & q > k_c. \end{cases} \tag{7}$$

Fortunately, the details of the way in which $\nu(q)$ levels off around $q = k_c$ only affects the determination of the constant Z and not that of η, which depends only on the small-q form of $\nu(q)$. This is a reflection of the fact that η is a universal (model-independent) function of n and d, whereas Z is not.[10]

To determine η we need the k-dependent part of the self-energy: From Eq. (2)

$$\sigma(k) - \sigma(0) = (2/n)\textstyle\sum_q \nu(q)[g(\vec{k}+\vec{q}) - g(q)].$$

Using Eqs. (5) and (7) gives[9]

$$\sigma(k) - \sigma(0) = (2/n) Z [g(d,\eta)/f(d,\eta)] k^2 (k_c/k)^\eta - (\text{term in } k^2), \quad 2 > \eta > 0, \tag{8}$$

where the term in k^2 comes from splitting the sum at $q = k_c$ in accordance with Eq. (7), and where

$$g(d,\eta) = \frac{1}{(4\pi)^{d/2}} \frac{2}{\eta}\left(\frac{4-d-2\eta}{2-\eta}\right) \frac{\Gamma(1+\frac{1}{2}\eta)\Gamma(2-\eta)\Gamma(\frac{1}{2}\eta+\frac{1}{2}d-1)}{\Gamma(1-\frac{1}{2}\eta)\Gamma(\eta+\frac{1}{2}d-1)\Gamma(1+\frac{1}{2}d-\frac{1}{2}\eta)}.$$

We can now find η by noting that $\sigma(k) - \sigma(0)$ is also given by Eqs. (1) and (5):

$$\sigma(k) - \sigma(0) = Z k^2 (k_c/k)^\eta - k^2. \tag{9}$$

Matching of the terms in k^2 in Eqs. (8) and (9) determines the constant Z (subject to the remarks above).

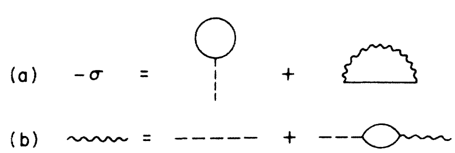

FIG. 1. (a) Self-energy graphs in the SCSA. (b) Dyson equation for the screened potential.

Matching of the terms in $k_c^{\eta}k^{2-\eta}$ determines η via the condition

$$f(d,\eta)=(2/n)g(d,\eta),$$

giving

$$\eta=\frac{4}{n}\left(\frac{4-d-2\eta}{2-\eta}\right)\frac{\Gamma(1+\frac{1}{2}\eta)\Gamma(1-\frac{1}{2}\eta)\Gamma(2-\eta)\Gamma(d+\eta-2)}{\Gamma(\eta+\frac{1}{2}d-1)\Gamma(1+\frac{1}{2}d-\frac{1}{2}\eta)\Gamma(2-\eta-\frac{1}{2}d)\Gamma(\frac{1}{2}d+\frac{1}{2}\eta-1)}+O(1/n^2). \tag{10}$$

I call the solution of Eq. (10) $\eta_{sc}(n,d)$. Some special cases are of particular interest.

(i) *The limit* $n\to\infty$.—In the large-n limit η becomes small enabling us to replace it, to lowest order, by zero in the right-hand side of Eq. (10). This gives

$$\eta_{sc}=\frac{4}{n}\left(\frac{4-d}{d}\right)\frac{\Gamma(d-2)}{\Gamma(2-\frac{1}{2}d)\Gamma(\frac{1}{2}d)[\Gamma(\frac{1}{2}d-1)]^2}, \quad n\gg 1, \tag{11}$$

in agreement with the results of Abe and Hikami[1] and of Ma.[4] As expected, our result is exact to order $1/n$.

(ii) *The case* $d=3$.—Setting $d=3$ in Eq. (10) yields

$$\eta=\frac{16}{n\pi^2}\,\frac{1-2\eta}{(2-\eta)(3-\eta)}\,\frac{\pi\eta}{\tan(\pi\eta)}\,\frac{\pi\eta/2}{\tan(\pi\eta/2)}. \tag{12}$$

In the large-n limit one recovers the standard result[3]

$$\eta_{sc}\simeq\frac{8}{3\pi^2 n}, \quad n\gg 1. \tag{13}$$

Expanding to order $1/n^2$ gives

$$\eta_{sc}=(8/3\pi^2 n)-\tfrac{7}{6}(\tfrac{8}{3})^2(1/\pi^4 n^2)+O(n^{-3}). \tag{14}$$

This should be compared to Abe's exact result[2] to order $1/n^2$:

$$\eta_{Abe}=(8/3\pi^2 n)-(\tfrac{8}{3})^3(1/\pi^4 n^2). \tag{15}$$

We see that the SCSA underestimates the coefficient of the $1/n^2$ term by a factor $\frac{7}{16}$. In fact it is interesting to note that Eq. (14) follows directly from including only graphs (a) and (c) of Abe's calculation.[2] This is to be expected since these graphs are nonskeleton, i.e., they result from making propagator self-energy insertions in lower-order graphs. Abe's graphs (b) and (d), on the other hand, are skeleton graphs and are therefore not included in the present calculation.

Equation (12) can be solved numerically for $\eta_{sc}(n,3)$. The result is shown in Fig. 2. Also plotted are the first-order result, Eq. (13), and Abe's result, Eq. (15). Particular values are $\eta_{sc}(3,3)\simeq 0.079$ and $\eta_{sc}(1,3)\simeq 0.177$. High-temperature-series estimates[11] are somewhat lower: $\eta(3,3)\simeq 0.043\pm 0.014$ and $\eta(1,3)\simeq 0.055\pm 0.01$. I conclude that the SCSA is unreliable for $n\lesssim 1$ as is to be expected of a large-n approximation. Note also that the η versus n curve should eventually bend over so that η vanishes[12] at $n=-2$.

(iii) *The case* $d=2$.—This case provides the most interesting application of the self-consistent method since the simple $1/n$ expansion breaks down here by virtue of the spherical model exhibiting no phase transition. One may attempt to circumvent this problem by performing the expansion for arbitrary d ($2<d<4$) and taking the limit $d\to 2+$ at the end. In first order this gives $\eta=\epsilon/n$, $\epsilon=d-2\ll 1$ [as may be seen by setting $d=2+\epsilon$ in Eq. (11)], leading to a vanishing η as $\epsilon\to 0$. It has been conjectured,[13] however, that this procedure will produce a nonzero result in second order. The present calculation does not support this view. Expanding our general result to order n^{-2} for $d=2+\epsilon$, yields $\eta_{sc}=\epsilon/n+2\epsilon/n^2+O(n^{-3})$, i.e., the nonskeleton second-order graphs vanish linearly as $d\to 2+$. Work in progress[14] indicates that this is also true of the skeleton graphs, not included here.

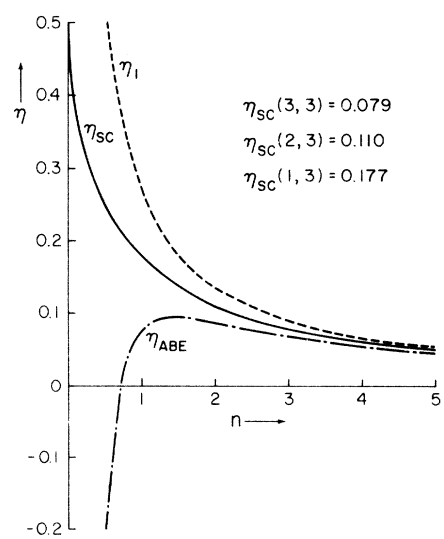

FIG. 2. The exponent η versus n for $d=3$. The SCSA, order $1/n$, and order $1/n^2$ results are labeled η_{sc}, η_1, and η_{Abe}, respectively.

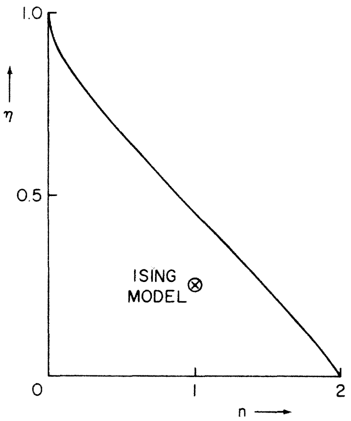

FIG. 3. The exponent η versus n for $d=2$ within the SCSA. The exact solution of the two-dimensional Ising model, $\eta(1,2)=\frac{1}{4}$, is shown for reference.

Setting $d=2$ in Eq. (10) yields

$$\eta=\frac{8\eta}{n}\frac{(1-\eta)^2}{(2-\eta)^2}+O(1/n^2),$$

giving either the trivial solution $\eta_{sc}=0$ or the nontrivial result

$$\eta_{sc}=(2/8-n)[4-n\pm(2n)^{1/2}].$$

where the negative root must be taken to satisfy the conditions on η in Eq. (6). The result is shown in Fig. 3. For $n>2$ one must take the trivial root $\eta_{sc}=0$. The Ising-model result is shown for comparison. Since SCSA is essentially a large-n approximation one should not place too much faith in the numerical values it predicts for these small-n values. I conjecture that including higher-order terms in $1/n$ exactly will merely shift the value of n at which η vanishes. [Recent work by Kosterlitz and Thouless[15] predicts a transition for $n=2$ with $\eta(2,2)=\frac{1}{4}$ but no transition for $n=3$ and therefore presumably $\eta(3,2)=0$.] Here I assume that the vanishing of η at each order in $1/n$ for $d=2+$, which holds in the present model, is a general result. Note however, that the forms $\eta\sim 1/n^x$ (x nonintegral) and $\eta\sim e^{-n}$, for example, would lead to similar behavior and cannot be ruled out.

It is a pleasure to acknowledge helpful discussions with Professor R. A. Ferrell. The award of a Fulbright-Hays scholarship is also gratefully acknowledged.

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