Wiegand and Godfrey[6]) exhibit sharp intensity variations with Z which must be coming from variations in atomic structure. Since in the Fermi-Teller model individual atomic structure is abandoned in favor of the statistical model, it is only sensible to compare our results with the experimental intensities *smoothed over* Z. While this considerably weakens the test, we think that the agreement is fairly good. Our maximum yields are in the range (20–40)%; in contrast, Eisenberg and Kessler[2] and Martin,[8] because they used a "statistical" $P(l)$ [i.e., Eq. (1) with $a=0$], found maximum yields of about 70%.

Hence we conclude that, to the extent that one is willing to ignore effects due to individual atomic structure, the Fermi-Teller model provides an adequate framework for calculating initial capture and de-excitation. A detailed account of this work will appear elsewhere.

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†Visiting Staff Member, University of California, Los Alamos Scientific Laboratory.

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Temporal Electrostatic Instabilities in Inhomogeneous Plasmas*

Y. C. Lee and P. K. Kaw†
University of California, Los Angeles, California 90024
(Received 12 November 1973)

The parametric process where the electromagnetic wave drives two Langmuir oscillations near $2\omega_p$ can be shown to be temporally unstable even in inhomogeneous plasmas. The instability arises because of wave trapping which prevents the convection of wave energy out of the unstable region.

Recently there has been much interest in parametric instabilities in inhomogeneous plasmas which have applications both in controlled fusion experiments and in high-intensity laser-pellet interactions. Rosenbluth[1] presented a theory of parametric three-wave interactions within the WKB approximation. His results show that, for the case $k=k'(0)x$, instabilities can be saturated by spatial convection of wave energy out of the unstable region. Subsequent investigation by Drake and Lee[2] indicates the existence of temporal instabilities when the WKB approximation is no longer valid. In this paper we study the parametric decay instabilities of Langmuir oscillations at $\omega_0 \simeq 2\omega_p$. We find that both Langmuir waves can be trapped in the vicinity of their cutoffs, resulting in a strong temporal instability benefitted by the Airy enhancement of both unstable waves.

The basic equations governing the parametric instability of two Langmuir waves in an inhomogeneous plasma have been obtained by Drake and Lee.[2] Here we simply state the results:

$$\nabla\cdot\left[\frac{3a^2}{2\omega_p^2}\nabla\nabla\cdot+1-\frac{\omega_p^2}{\omega^2}+\frac{2i\nu}{\omega_p}\right]\vec{E}_+=\frac{2e}{m\omega_0^2}\nabla\cdot[\nabla(\vec{E}\cdot\vec{E}_{0+})-\vec{E}_{0+}\nabla\cdot\vec{E}_-],\tag{1}$$

$$\nabla\cdot\left[\frac{3a^2}{2\omega_p^2}\nabla\nabla\cdot+1-\frac{\omega_p^2}{\omega_-^2}-\frac{2i\nu}{\omega_p}\right]\vec{E}_-=\frac{2e}{m\omega_0^2}\nabla\cdot[\nabla(\vec{E}_+\cdot\vec{E}_{0-})-\vec{E}_0\nabla\cdot\vec{E}_+],\tag{2}$$

where $\omega_-=\omega-\omega_0$, and ν is the total damping rate of Langmuir waves (collisional damping plus Landau damping). We have also assumed a plane electromagnetic pump wave, $\vec{E}_0=\vec{E}_{0+}\exp(+i\omega_0 t-ik_0x)+\vec{E}_{0-}\times\exp(-i\omega_0t+ik_0x)$, propagating along the plasma density gradient (in the x direction), and two electrostatic perturbations $\vec{E}_+(x)\exp(i\omega t-ik_\perp y)$ and $\vec{E}_-(x)\exp(i\omega_-t-ik_\perp y)$. Approximating the plasma density by a linear profile in the vicinity of the linear cutoff $\omega_p=(\omega_0^2/4-3a^2k_\perp^2/2)^{1/2}$ and keeping only the lowest order terms in $k_0/k_\perp$, we rewrite Eqs. (1) and (2) as

$$\frac{d}{dx}\left(\frac{d^2}{dx^2}-\beta x+i\Gamma\right)\frac{d}{dx}\varphi_+-k_\perp^2\left(\frac{d^2}{dx^2}-\beta x+i\Gamma\right)\varphi_+=A\frac{d}{dx}\varphi_-,\tag{3}$$

$$\frac{d}{dx}\left(\frac{d^2}{dx^2}-\beta x-i\Gamma\right)\frac{d}{dx}\varphi_--k_\perp^2\left(\frac{d^2}{dx^2}-\beta x-i\Gamma\right)\varphi_-=-A\frac{d}{dx}\varphi_+,\tag{4}$$

where $\vec{E}_\pm=\nabla\varphi_\pm$, $\beta=2\omega_p^2/3a^2L$, $\Gamma=(8\omega_p^2/3a^2)(\nu+\gamma)/\omega_0$, and $A=\frac{8}{3}k_0k_\perp(\omega_p^2/a^2)eE_0/m\omega_0^2$. γ is the growth rate of the Langmuir waves and L is the density scale length.

Equations (3) and (4), as they stand, contain four parameters. They are not all independent, however, as can be seen by the following change of variables:

$$z=\beta^{1/3}x,\quad k^2=\beta^{-2/3}k_\perp^2,\quad \sigma=\beta^{-2/3}\Gamma,\quad M=A\beta^{-1},\tag{5}$$

Equations (3) and (4) now become

$$\left[\left(\frac{d^2}{dz^2}-k^2\right)\left(\frac{d^2}{dz^2}+i\sigma-z\right)+\frac{d}{dz}\right]\varphi_+=M\frac{d}{dz}\varphi_-,\tag{6}$$

$$\left[\left(\frac{d^2}{dz^2}-k^2\right)\left(\frac{d^2}{dz^2}-i\sigma-z\right)+\frac{d}{dz}\right]\varphi_-=-M\frac{d}{dz}\varphi_+.\tag{7}$$

These equations can be solved by treating the pump field as well as σ (damping plus growth rate) as perturbations. To lowest order we have

$$\left[\left(\frac{d^2}{dz^2}-k^2\right)\left(\frac{d^2}{dz^2}-z\right)+\frac{d}{dz}\right]\varphi_\pm^{(0)}=0,\tag{8}$$

which differs from the usual linear turning-point (Airy) equation only in the presence of the last term, $d\varphi_\pm/dz$, which arises from the charge bunching produced by particle motion along the density gradient. Such an effect is often neglected in the weak-inhomogeneity case. Equation (8) has an integral solution of the form

$$\varphi_+^{(0)}(z)=\int_{-i\infty-c}^{i\infty-c}dt\,\exp[zt-\tfrac{1}{3}t^3-\tfrac{1}{2}\ln(t^2-k^2)],\tag{9}$$

where $c>0$. The logarithms in the t plane are defined by the branch cuts; one runs from $t=k$ to ∞, the other from $t=-k$ to $-\infty$. Asymptotically, $\varphi_\pm^{(0)}(z)$ differ significantly from the Airy solution only for $z\gg0$, where the $\varphi_\pm^{(0)}(z)$ behave like $e^{-kz}/\sqrt{z}$.

Knowing the zeroth-order solutions $\varphi_\pm^{(0)}(z)$, we now seek solutions to Eqs. (6) and (7) of the form

$$\varphi_\pm(z)=\int_{-i\infty-c}^{i\infty-c}dt\,\exp[zt-\tfrac{1}{3}t^3-\tfrac{1}{2}\ln(t^2-k^2)]f_\pm(t).\tag{10}$$

The equation that governs $f_+(t)$ can be obtained by substituting (10) into Eqs. (6) and (7) and eliminating

$f_-(t)$. The equation that results is

$$\frac{d^2}{ds^2}f_+(is)+\left\{M^2\frac{s^2}{(s^2+k^2)^2}-\sigma^2+\sigma\left[\frac{d}{ds}g(s)\right]-\frac{1}{4}\left(\frac{d}{ds}g\right)^2+\frac{1}{2}\left(\frac{d^2}{ds^2}g\right)\right\}f_+(is)=0, \tag{11}$$

where $s=-it$ and $g(s)=\ln[s/(s^2+k^2)]$. In Eq. (11), we can get rid of the terms involving $g(s)$ by assuming $k^2 \gg 1$ [i.e., the perpendicular wavelength $2\pi/k_\perp$ being much smaller than the characteristic Airy scale length $(L/k_D^2)^{1/3}$]. The above condition also ensures that $f_+(t)$ is a slowly varying function in t (or is) provided that both M^2/k^2 and σ^2 are at most of order unity.

$f_+(is)$ being a slowly varying function in s allows us to evaluate $\varphi_+(z)$ in (10) for large negative values of z by the method of steepest descent. The result is

$$\varphi_+(z)=(|z|+k^2)^{-1/2}\sqrt{\pi}\,|z|^{-1/4}\{\exp(3i\pi/4)\exp[(2i/3)|z|^{3/2}]f_+(i|z|^{1/2})+\exp(i\pi/4)\exp[-(2i/3)|z|^{3/2}]f_+(-i|z|^{1/2})\}. \tag{12}$$

Boundary conditions for $\varphi_+(z)$ require $|z|^{-3/4}|f_+(\pm i|z|^{1/2})|$ to vanish with increasing $|z|$ for $z<0$. However, $f_+(is)$ in general tends toward $e^{\pm\sigma s}$ as $|s|\to\infty$. Given M and k^2, we therefore have an eigenvalue problem for σ^2. The WKB approximation together with the linear–turning-point techniques allows us to solve Eq. (11) for $M^2/k^2-\sigma^2\gg 1$, yielding the eigenvalue conditions:

$$\int_{s_1}^{s_2} ds[M^2s^2/(s^2+k^2)^2-\sigma_n^2]^{1/2}=(n+\tfrac{1}{2})\pi, \quad n=1,2,\ldots, \tag{13}$$

where $s_2>s_1>0$ are two linear turning points at which the integrand vanishes. From (13) we see that higher eigenmodes have lower growth rates (or larger damping). The maximum growth rates obviously occur when $|M^2/k^2-\sigma^2|\ll\sigma^2$, in which case the WKB approximation is not valid unless $M^2/k^2\gg 1$. Note, however, that around the trapping region (in s space!) we can approximate Eq. (11) by expanding $M^2s^2/(s^2+k^2)^2$ around $s=\pm k$. The equation that results is just the harmonic oscillator equation, which yields the following eigenvalues for the growth rate:

$$\frac{\gamma_n}{\omega_0}=-\frac{\nu}{\omega_0}+\frac{E_0}{4(4\pi n_0T_e)^{1/2}}k_0\lambda_D\left(1-\frac{n+\frac{1}{2}}{M}\right)^{1/2}, \quad n=0,1,2,\ldots. \tag{14}$$

The homogeneous threshold is recovered if we let $M\propto L\to\infty$. When the plasma is significantly inhomogeneous, M must exceed $n=\frac{1}{2}$ in order to have an instability. This gives the inhomogeneous threshold condition

$$E_0^2/4\pi n_0T_e>(n+\tfrac{1}{2})^2(k_0L)^{-2}(k_\perp\lambda_D)^{-2}, \quad n=0,1,2,\ldots. \tag{15}$$

Note that because of the assumption $k_\perp\gg k_0$, which rules out an instability with long wavelength, (15) should be regarded as a sufficient condition for instability but not necessary. Equation (15) can be more simply expressed in terms of the zeroth-order electron oscillating velocity, $V_0=(c/4)(k_\perp L)^{-1}$, with c being the speed of light. In comparison, the Raman $2\omega_p$ instability has a much higher threshold, $V_0=0.52c(k_0L)^{-2/3}$. However, once the thresholds are exceeded, both instabilities grow at nearly the same growth rate, $\gamma=(V_0/c)\omega_{pe}/2$, which is also the growth rate for both instabilities in a homogeneous plasma.

The eigenamplitude can also be obtained for $|(2k_D^2/3k_\perp^2L)x+1|\ll 1$ as follows:

$$\varphi_{+n}(x)=A_i((2k_D^2/3L)^{1/3}x)\exp\{-\tfrac{4}{3}(k_D^2V_0/k_\perp c)[3k_\perp^2L/2k_D^2)-x]\}\times H_n((8k_D^2V_0/3k_\perp c)^{1/2}(3k_\perp^2L/2k_D^2-x)^{1/2}), \quad n=0,1,2,\ldots, \tag{16}$$

where $A_i(\xi)$ and the $H_n(\xi)$ are the Airy function and Hermite polynomials, respectively. Equation (16) shows that the usual Airy behavior of the electron plasma waves are modified by a 100% modulation of its envelope. Such a phenomena should be easily detectable in the laboratory.

Hindsight allows us to interpret the threshold condition (15) in a simple way. In a homogeneous plasma the marginal stability condition including frequency mismatch has the approximate form

$$[\nu^2+(\Delta\omega)^2]^{1/2}/\omega_p=(V_0/c)k_xk_\perp/K^2,$$

where $K^2=k_x^2+k_\perp^2$, and $\Delta\omega$ is the frequency mismatch. Assume now that the homogeneous condition can be generalized to the inhomogeneous case with $\Delta\omega$ being replaced by the average mismatch due to density inhomogeneity. Clearly, if l is the characteristic wavelength of the unstable waves, then $\langle\Delta\omega\rangle$ should be approximately given by $(\lambda_D/l)^2\omega_p$. However, from hindsight we see that l is nothing but the typical Airy scale length $(L/k_D^2)^{1/3}$. This together with the generalized marginal stability condition immediately gives $V_0/c \simeq (k_\perp L)^{-1}$, provided that $k_\perp l \ll 1$. The last condition is the same as $k^2 \gg 1$ which we used in deriving Eq. (12). The above simple argument seems to be sufficiently general for various temporal instabilities in an inhomogeneous plasma and should deserve further investigations.

The electrostatic $2\omega_{pe}$ instability clearly would appear to be an important nonlinear absorption mechanism for plasma heating both in controlled thermonuclear reactions and in laser-pellet experiments because of its low threshold and large growth rate. It is especially attractive in the sense that the unstable products will eventually be locally absorbed by the plasma near the cutoff because of strong Landau damping in the underdense region.

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†Permanent address: Physical Research Laboratory, Navrangpura, Ahmedabad 9, India.

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Turbulence Spectrum Observed in a Collision-Free θ-Pinch Plasma by CO_2 Laser Scattering

N. L. Bretz and A. W. DeSilva*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742
(Received 10 May 1973)

Turbulence generated by a rapidly imploding current sheath in a high-voltage θ pinch has been investigated by CO_2 laser scattering and by emission spectroscopy of forbidden-line satellites in helium. Waves indicative of a saturated ion-sound instability are seen in the current sheath. In a region well behind the sheath, waves are seen with wave numbers near k_D. These waves may be the result of mode coupling from short-wave-number turbulence generated earlier by the passage of the current sheath.

Plasma-heating turbulence has been seen in several laboratory experiments featuring the application of rapidly rising current pulses to low-density ($n_e = 10^{13}$ cm^{-3}) plasmas.[1] Plasma heating results from instabilities of the current stream which transfer energy to plasma waves followed by decay via mode coupling and damping into thermal energy. Several instabilities have been proposed as mechanisms by which heating may occur.[2-5] We report here the use of scattering of CO_2 laser radiation (10.6 μm) from a low-β [$\beta \equiv nkT(B^2/8\pi)^{-1}$] turbulent θ-pinch plasma to measure the spectra of density fluctuations. The scattering is supplemented by measurements of the magnitude of the electric field fluctuations by helium satellite spectroscopy. These observations provide a direct way to study the properties of the instabilities.

The experiments were done in the Maryland fast θ pinch which has been described elsewhere.[6] Conditions just prior to the heating pulse are $n_e = 1.6\times10^{13}$ cm^{-3}, $T_e = 0.5$ eV, and $B_0 \doteq 175$ G. The magnetic field which excites the instabilities reaches 1500 G in 280 nsec, and may be either parallel or antiparallel to the bias field B_0. Measurements were attempted in hydrogen, deuterium, helium, and argon plasmas using both parallel and antiparallel configurations. The largest and most consistent scattered signals were seen in the parallel-bias helium case on which we report here.

The Debye wave number $k_D = (4\pi n_e e^2/kT_e)^{1/2}$ is determined at one radius ($r = 13$ cm) by measuring the electron temperature and density by Thomson scattering of ruby laser light. Because of a very high level of plasma radiation, the scattered light was collected at an angle of 20° to the incident beam direction by a lens immersed in the