introduced by multiple scattering.

The precision obtained in the intermediate-momentum range compares favorably with the one obtained in the small-momentum range. In an experiment based on Eq. (5) only one sample is needed and the result is independent of polydispersity. In contrast, the experimental determination of ν derived from relation (2) requires several samples, with well-defined molecular masses.

The value of ν obtained in good solvent in this experiment is consistent with results obtained using relation (2) and other experimental methods. The value of ν obtained in the bulk indicates the absence of an effective excluded-volume interaction over a large range of distances. A remarkable fact is that there is no deviation from the $1/q^2$ law for q values as high as 0.1 $\mathring{A}^{-1}$.

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Parametric Excitation of Alfvén and Ion Acoustic Waves

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The threshold is derived for parametric excitation of Alfvén waves in a uniform plasma in which the background magnetic field is modulated sinusoidally in time. Including the plasma pressure we show that ion acoustic waves can also be excited directly. Both cases are absolutely unstable for any interaction length. Finally, we analyze the subsequent decay of the excited Alfvén wave into an ion acoustic and another Alfvén wave.

The growth rate for the parametric excitation of Alfvén waves was calculated by Vahala and Montgomery.[1] The Alfvén waves were excited by oscillating the background magnetic field. This phenomenon was later observed experimentally by Lehane and Paoloni.[2] The purpose of this Letter is threefold; first, we derive the Alfvén wave instability threshold; second, we show that by taking the plasma pressure into account not only Alfvén waves may be excited directly but also ion acoustic waves; third, we obtain the threshold for the subsequent decay of the finite amplitude Alfvén wave so excited into an ion acoustic wave and another Alfvén wave.

We shall follow Vahala and Montgomery[1] and use simple magnetohydrodynamic equations with the addition of a plasma pressure term and resistivity. These equations are the following:

$$\rho\, \partial\vec{v}/\partial t + \rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla p + \mu_0^{-1}(\nabla\times\vec{B})\times\vec{B}; \tag{1}$$

$$\partial\vec{B}/\partial t = \nabla\times(\vec{v}\times\vec{B}) - (\eta/\mu_0)\nabla\times(\nabla\times\vec{B}); \tag{2}$$

$$\partial\rho/\partial t + \nabla\cdot(\rho\vec{v}) = 0; \tag{3}$$

we take the equation of state to be $p = c_s^2\rho$.

The model considered by Vahala and Montgomery[1] was that of a uniform column of plasma of radius a with a uniform magnetic field pointing in the z direction. The source of energy for the parametric excitation was then a perturbation to the background magnetic field which was approximately uniform spatially, but periodic in time. This perturbation to the background magnetic field then formed the new "oscillating equilibrium" whose stability properties were then examined. Vahala and Montgomery constructed the following equilibrium:

$$\vec{B}_0 = \hat{i}_z B_0[1 + \epsilon \operatorname{Re}\exp(-i\omega_0 t)],$$

$$\rho_0 = \rho_0[1 + \epsilon \operatorname{Re}\exp(-i\omega_0 t)], \quad v_0 = 0,$$

where ω_0 is the modulation frequency and ϵ is a small parameter measuring the percentage modulation of the uniform magnetic field and mean plasma density. This zero-order (or oscillating equilibrium) solution is not affected by the inclusion of plasma pressure. It is also unaffected by the resistivity term provided $\nu/\omega_0 \lesssim \omega_{pe}^2 a^2/c^2$, where ν is the electron-ion collision frequency and c and ω_{pe} have their usual meaning.

Let us now calculate the threshold for the excitation of Alfvén waves by the modulation of the equilibrium state. The ordering chosen by Vahala and Montgomery allows one to consider Alfvén waves which propagate only in the direction of $\vec{B}_0$ and whose wavelength is long compared with the plasma radius. For simplicity we consider waves polarized such that $\vec{B}_1 = (B_x, 0, 0)$. Now taking the x components of Eqs. (1) and (2) we obtain

$$\rho_0 \frac{\partial v_x}{\partial t} - \frac{B_0}{\mu_0}\frac{\partial B_x}{\partial_z} = -\epsilon\rho_0 \operatorname{Re}[\exp(-i\omega_0 t)]\frac{\partial v_x}{\partial t} + \frac{\epsilon B_0}{\mu_0}\operatorname{Re}[\exp(-i\omega_0 t)]\frac{\partial B_x}{\partial z}, \tag{4}$$

$$\frac{\partial B_x}{\partial t} - \frac{\eta}{\mu_0}\frac{\partial^2 B_x}{\partial z^2} - B_0\frac{\partial v_x}{\partial z} = \epsilon B_0 \operatorname{Re}[\exp(-i\omega_0 t)]\frac{\partial v_x}{\partial z}, \tag{5}$$

where we have included only those terms which couple the modulation to the Alfvén waves.

We now put Eqs. (4) and (5) into the coupled-mode[3,4] form using the selection rules $\omega_0 \approx \omega_1 + \omega_2$ and $k_0 = k_1 + k_2$, where $k_0 = 0$ and (ω_1, k_1) and (ω_2, k_2) are the frequency and wave number of the two Alfvén waves we are investigating.

The Alfvén wave solution B_1 varies approximately as $\operatorname{Re}\exp[i(k_1 z - \omega_1 t)]$, where $\omega_1 = |k_1|C_A$ and C_A is the Alfvén speed. The wave equation obtained from Eqs. (4) and (5) describing the coupling of this wave to the second Alfvén wave B_2 and the modulation is

$$\left(\frac{\partial}{\partial t} + \frac{k_1^2\eta}{2\mu_0}\right)b_1 = i\frac{\epsilon}{4}\frac{k_2^2 C_A^2}{\omega_2} b_2^* e^{-i\varphi t}, \tag{6}$$

where $b_1(t)$ is a slowly varying amplitude (compared with ω_0) given by $B_1(z,t) = b_1(t)\exp[i(k_1 z - \omega_1 t)]$ and $\varphi \equiv \omega_0 - \omega_1 - \omega_2$. Similarly, the equation for the second Alfvén wave, obtained from Eqs. (4) and (5), is

$$\left(\frac{\partial}{\partial t} + \frac{k_2^2\eta}{2\mu_0}\right)b_2 = i\frac{\epsilon}{4}\frac{k_1^2 C_A^2}{\omega_1} b_1^* e^{-i\varphi t}, \tag{7}$$

where $b_2(t)$ is the slowly varying amplitude of wave B_2 defined as for wave B_1. (Note that $\omega_2 \equiv |k_2|C_A$ and $k_2 = -k_1$, from the matching conditions.) Equations (6) and (7) yield the dispersion relation

$$(\omega - \varphi + i\gamma_A)(\omega + i\gamma_A) + (\tfrac{1}{4}\epsilon)^2\omega_1\omega_2 = 0, \tag{8}$$

where $\gamma_A \equiv k_1^2\eta/2\mu_0$. This equation has an unstable solution when the amplitude of the modulation exceeds a threshold value given by

$$(\tfrac{1}{4}\epsilon)^2(\tfrac{1}{2}\omega_0)^2 = \gamma_A^2 + (\tfrac{1}{2}\varphi)^2, \tag{9}$$

where we have used the fact that $\omega_1 = \omega_2 \approx \omega_0/2$. The minimum threshold occurs for perfect matching ($\varphi = 0$) and is

$$\epsilon_{\min} = \omega_0\eta/\mu_0 C_A^2. \tag{10}$$

Well above this threshold the growth rate of the two excited Alfvén waves is

$$\gamma = \epsilon\omega_0/8, \tag{11}$$

which is the result obtained by Vahala and Montgomery.[1]

Now consider the second possibility, i.e., the excitation of ion acoustic waves by the modulation of the equilibrium state. In this case, it is the modulation of the average density which couples to the acoustic waves. Again we consider waves which propagate along the uniform magnetic field. The wavelength of the excited acoustic waves depends on the value of β ($= 2\mu_0 n_0 \kappa T_e/B_0^2$). The ion acoustic wave number k_s must satisfy $k_s a \sim \delta\beta^{-1/2}$, where δ is $a\omega_0/C_A$ and has been assumed to be $O(\epsilon^{1/2})$.

The equations required to describe this process are the z component of Eqs. (1) and (3). [N.B.: We have added a phenomenological collision term to the z component of Eq. (1) in order to simulate the damping of the ion acoustic mode.]

We now proceed as for the previous case using v_z as the wave amplitude and calculating the perturbation to the two acoustic waves which vary as $v_{z1} \sim \mathrm{Re}\exp[i(k_{s1}z - \omega_{s1}t)]$ and $v_{z2} \sim \mathrm{Re}\exp[i(k_{s2}z - \omega_{s2}t)]$, where $\omega_{s1,2} \equiv |k_{s1,2}|c_s$. The resulting coupled-mode equations are

$$\left(\frac{\partial}{\partial t}+\frac{\nu}{2}\right)V_1 = -i\frac{\epsilon}{4}\frac{\omega_0\omega_{s2}}{\omega_{s1}}V_2^*\exp(-i\varphi_s t), \qquad (12)$$

$$\left(\frac{\partial}{\partial t}+\frac{\nu}{2}\right)V_2 = -i\frac{\epsilon}{4}\frac{\omega_0\omega_{s1}}{\omega_{s2}}V_1^*\exp(-i\varphi_s t), \qquad (13)$$

where $V_{1,2}$ are the slowly varying amplitudes of the acoustic waves $v_{z1,2}$ defined as for the previous case, and $\varphi_s \equiv \omega_0 - \omega_{s1} - \omega_{s2}$, where we have again used the selection rules $\omega_0 \approx \omega_{s1} + \omega_{s2}$ and $0 = k_{s1} + k_{s2}$; ν is the phenomenological collision frequency. (It is expected that the main damping mechanism of the ion acoustic waves will be ion and electron Landau damping.) The analysis of Eqs. (12) and (13) proceeds as before giving as the threshold for instability

$$(\epsilon/4)^2\omega_0^2 = \gamma_s^2 + (\varphi_s/2)^2, \qquad (14)$$

where $\gamma_s \equiv \nu/2$. The minimum threshold is again, for perfect matching ($\varphi_s = 0$),

$$\epsilon_{\min} = 4\gamma_s/\omega_0. \qquad (15)$$

The growth rate of the two ion acoustic waves (one traveling in the $+z$ direction the other in the $-z$ direction) well above threshold is given by

$$\gamma = (\epsilon/4)\omega_0. \qquad (16)$$

This process excites ion acoustic waves of frequency $\omega_0/2$ (the same as the Alfvén waves) but usually with a much shorter wavelength than the Alfvén waves, unless $\beta \sim O(1)$.

In other parametric instabilities, where ion acoustic waves are excited, only a small fraction of the pump energy goes to the acoustic wave (because of the Manley-Rowe relations). However, in the direct excitation we have described the energy from the pump flows entirely to the acoustic waves. (The competition for the pump energy would be between different parametric instabilities rather than the decay products of one instability.) In view of this, the instability described above could be an important plasma heating mechanism.

We now describe the subsequent decay of the excited Alfvén waves into another Alfvén wave and an ion acoustic wave. For a low-β plasma this process produces a long-wavelength, low-frequency, ion acoustic wave, compared with the ion acoustic wave excited directly. The decay of an Alfvén wave in the manner described was first analyzed several years ago by Galeev and Oraevskii.[5] However, they gave only the initial growth rate whereas we shall calculate the threshold both for the decay instability and a purely growing (or modified decay[6]) instability which is a magnetic analog of the oscillating two-stream mode.[7]

We again obtain coupled-mode equations for the nonlinear interaction of Alfvén and ion acoustic waves, giving the dispersion relation

$$(\Omega+\delta+i\gamma_2)(\Omega-\delta+i\gamma_2)(\Omega^2-\omega_s^2+2i\gamma_s\Omega)+K\delta/\omega_2 = 0, \qquad (17)$$

where

$$K \equiv k_1|k_2|\,k_s^2C_A^2|b_1|^2/4\rho_0\mu_0, \quad \gamma_2 \equiv k_2^2\eta/2\mu_0, \quad \delta \equiv \omega_1 - \omega_2, \quad \Omega \equiv \omega - \delta,$$

and where we have taken the usual selection rules. In order to satisfy these relations for Alfvén and ion acoustic waves we have taken $k_1 > 0$, $k_s > 0$, $k_2 < 0$; and $\omega_{1,2,s}$ are all positive. The amplitude of the Alfvén wave (a standing wave) excited by the modulation of the equilibrium state is denoted by b_1. We have therefore taken our pump wave b_1 to be

$$b_1(z,t) = \mathrm{Re}\{b_1\exp[i(k_1z-\omega_1t)] + b_1\exp[i(k_1z+\omega_1t)]\};$$

ω_2 and k_2 are the respective frequency and wave number of the excited Alfvén wave and γ_2 is its damping factor. Equation (17) is exactly the form of Nishikawa's model dispersion relation.[7] Using his result we can immediately write down the minimum thresholds for the purely growing and decay instabilities, respectively,

$$K_m = 2\omega_s^2\omega_2\gamma_2 \qquad (18)$$

and

$$K_m = 4\omega_s\omega_2\gamma_s\gamma_2(1-\gamma_s^2/4\omega_s^2), \qquad (19)$$

where Eq. (19) is only valid for $\gamma_2 \ll \omega_s$. The threshold for the decay mode is lower than that for the purely growing or modified decay instability.

The threshold for the decay of the excited Alfvén wave can be written in terms of $|b_1|/B_0$. If we compare the threshold value of this quantity with the minimum threshold for the initial excitation of the standing Alfvén wave, we obtain

$$|b_1|/\epsilon B_0 \sim (c_s \gamma_s / C_A \gamma_2)^{1/2}.$$

If $\beta \ll \gamma_2/\gamma_s$, then the excited Alfvén wave will decay at a much lower value of $|b_1|/B_0$ than the initial amplitude of modulation. In this case the decay of the Alfvén wave into another Alfvén wave and an ion acoustic wave will be an important saturation mechanism for the original Alfvén instability. Comparing the threshold values of ϵ required for the excitation of Alfvén waves and the direct excitation of ion acoustic waves, Eqs. (10) and (15) show that the Alfvén threshold is much lower. However, if ϵ is well above both thresholds then we can see from Eqs. (11) and (16) that the ion acoustic growth rate is double that of the Alfvén instability.

We have also considered the spatial dependence of these two instabilities. In both cases the two excited waves travel in opposite directions and both instabilities are absolute. Also, in both cases, the pair of excited waves have equal and opposite group velocities. If we follow the analysis of Kroll,[8] it turns out that both cases are absolutely unstable for any value of the interaction length L, i.e., **there is no critical length**! (N.B.: This result is only true for a uniform plasma.)

Summarizing the main results of this Letter we have shown that by modulating the background magnetic field sinusoidally in time both Alfvén and ion acoustic waves can be excited. We have calculated the threshold values for the percentage modulation of the magnetic field for these two cases. For a low-β plasma the ion acoustic waves excited will have very short wavelength in comparison with the Alfvén waves. In addition, the ion acoustic instability may be an effective method for heating a plasma since the pump energy flows entirely to the acoustic waves. Provided $T_e \gg T_i$ in the experiment of Lehane and Paoloni,[2] the acoustic waves should have been produced in their experiment. Both the Alfvén instability and the ion acoustic instability have been shown to be absolutely unstable. In both cases there was no critical length. Finally, we have calculated the threshold value of the amplitude of the Alfvén wave excited for further decay into another Alfvén wave and an ion acoustic wave. The ion acoustic wave excited in this decay has a wavelength comparable to the Alfvén wave.

An extension of this work to two dimensions will be forthcoming. This will be more readily comparable with experiment and may also be of relevance to experiments on transit time magnetic pumping.

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