How to Test Scaling in Asymptotically Free Theories*

David J. Gross†
Rockefeller University, New York, New York 10021
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It is shown how, in asymptotically free gauge theories, one can, given the deep-inelastic structure functions for values of q^2 in the asymptotic region, calculate explicitly the structure functions for all larger q^2. The ratio of structure functions for $q^2=50$ and 5 GeV^2 is estimated. Substantial deviations from scaling are found, as well as a dramatic ω dependence, for values of $1 \leqslant \omega \leqslant 4$.

Asymptotically free gauge theories[1,2] of the strong interactions predict a logarithmic decrease for the moments of the deep-inelastic structure functions.[1-3] These predictions are of the form ($x=q^2/2\nu$)

$$M_N(q^2)=\int_0^1 dx\, x^N F_2(x,q^2) \underset{q^2\to\infty}{\simeq} Z_N(\bar{g}^2)F_N(\bar{g}^2), \tag{1}$$

where $\bar{g}(q^2)$ is the "effective coupling constant" which vanishes for large q^2. For small $\bar{g}^2$, Z_N is determined by the anomalous dimension γ_N of the relevant twist-2 operator of spin $N+2$:

$$Z_N(\bar{g}^2)=\exp[-\int_g^{\bar{g}}\gamma_N(x)\,dx/\beta(x)] \underset{\bar{g}\to 0}{\simeq} (\bar{g}^2/g^2)^{A_N}[C_N+O(\bar{g}^2)], \tag{2}$$

where $\beta(x)=-\frac{1}{2}b_0x^3+O(x^5)$ is the standard renormalization group function. A_N is determined by the second-order contribution to $\gamma_N(x)=\gamma_N x^2+O(x^4)$: $A_N=b_0^{-1}\gamma^N$. The function $F_N(\bar{g}^2)$ will be determined by the unknown matrix element of the relevant operator, as well as by its Wilson coefficient. As $\bar{g}^2$ vanishes it approaches some, unknown constant:

$$F_N(\bar{g}^2)=F_N+O(\bar{g}^2). \tag{3}$$

As emphasized by Gross and Wilczek,[4] and more emphatically by Politzer,[5] the approach to the asymptotic region is controlled by the behavior of $\bar{g}^2(q^2)$. When this is sufficiently small the asymptotic form for M_N will be valid up to terms of order $\bar{g}^2$ itself. Also some sum rules should be satisfied up to terms of order $\bar{g}^2$. In particular the ratio of longitudinal to transverse moments should be of order $\bar{g}^2$. In this region $\bar{g}^2$ will behave as

$$\bar{g}^2 \underset{q^2\to\infty}{\simeq} 2/b_0 t, \tag{4}$$

where $t=\ln(q^2/\mu^2)$, μ^2 being an unknown scale parameter.

In this region one can test these theories by measuring the q^2 dependence of the moments M_N:

$$M_N(q^2)=\left(\frac{\ln q'^2}{\ln q^2}\right)^{A_N} M_N(q'^2)[1+O(\bar{g}^2)]. \tag{5}$$

Once these relations can be tested they will provide extremely clean and critical tests of asymptotically free gauge theories. The only unknowns are the following:

(1) The region of q^2 at which asymptotic behavior sets in. This will be determined by the value of q^2 at which $\bar{g}^2$ is small enough. To get some idea of this we consider the popular red-white-blue quark model. The strong interaction gauge group is SU(3)′, and $b_0=9/8\pi^2$.[1] The effective expansion parameter in this model is then

$$\bar{g}^2/4\pi^2 \underset{q^2\to\infty}{\approx} 4/9t. \tag{6}$$

(2) The scale parameter μ. This is at present totally undetermined. If, however, we take $\mu=1$ GeV as a reasonable value, then the expansion parameter above equals 0.27 for $q^2=5\ \mathrm{GeV}^2$ and 0.11 for $q^2=50\ \mathrm{GeV}^2$. Thus one might hope to use Eq. (5) to compare $q'^2=5\ \mathrm{GeV}^2$ data with, say, $q^2=50\ \mathrm{GeV}^2$ data.[6]

(3) The gauge group of the strong interactions and the nature of the quark representations. After all, the arguments for the red-white-blue model are not entirely compelling. This affects b_0 as well as A_N. The coefficients A_N have been evaluated for all groups.[1,3,4] For the nonsinglet piece of the structure functions they are given by

$$A_N=G\left(1-\frac{2}{(N+2)(N+3)}+4\sum_{k=0}^{N}\frac{1}{k+2}\right), \tag{7}$$

where G is totally determined by the gauge group and the representation of the quarks.[4] If the group is SU(3)′, and the theory contains three quark triplets, then $G=\frac{4}{27}$. Increasing the number of triplets will increase G and thus lead to larger violations of scaling (if there are three more colored charmed quarks, $G=\frac{4}{25}$). Ultimately G should be determined by experiment. The singlet coefficients are essentially identical to Eq. (7), except that $A_0^{\rm singlet}=0$.

Unfortunately there is little hope of testing Eq. (5) directly in the near future, since this would require a determination of the structure functions for large q^2 at *all* values of x. Furthermore the theory itself predicts large cancelations of the q^2 dependence of the structure functions when one calculates its moments. This is because $F_2(x,q^2)$ will increase, with increasing q^2, for small x, and decrease for x near 1, so as to keep the area M_0 constant.

It would clearly be much more useful to have a direct relation between the structure functions themselves at different values of q^2. This can be achieved by constructing a function $T(t/t',x)$ whose Nth moment with respect to x yields $(t/t')^{-A_N}=(\ln q'^2/\ln q^2)^{A_N}$. Then Eq. (5) simply states that the Mellin transform of $F_2(x,q^2)$ is equal to the product of the Mellin transforms of $F_2(x,q'^2)$ and of $T(t/t',x)$. One then uses the convolution theorem of Mellin transforms[7] to derive (I find it easier here to use the variable $\omega=2\nu/q^2=x^{-1}$)

$$F_2(\omega,t)=\int_1^{\omega}\frac{d\omega}{\omega'}F_2\left(\frac{\omega}{\omega'},t'\right)T\left(\frac{t}{t'},\omega'\right), \tag{8}$$

where

$$T\left(\frac{t}{t'};\omega\right)=\frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty}ds\left(\frac{t}{t'}\right)^{-A_s}\omega^{s+1}, \tag{9}$$

with A_s the analytic continuation of A_N to $\mathrm{Re}s\geq 0$. [This analytic continuation is valid, in fact, for $\mathrm{Re}s>-1$ as long as $F_2(\omega,t)$ is bounded for large ω.]

In principle this allows one, given the function $F_2(\omega,t'=\ln q'^2/\mu^2)$ for q'^2 in the asymptotic region, to calculate the structure function for all larger q^2 at any value of ω. In practice this relation is made more useful by the fact that A_N is approximately given for large N by

$$A_N\simeq G\left[4\ln(N+2)-0.69+\frac{2}{N+2}-\frac{7}{3(N+2)^2}+\frac{2}{(N+2)^3}+\dots\right]. \tag{10}$$

The large-N behavior of A_N will determine the kernel T for small values of ω. Indeed, using the expansion (10), one easily derives[8]

$$T\left(\frac{t}{t'}\right)=\left(\frac{t}{t'}\right)^{0.69G}\frac{(\ln\omega)^{P-1}}{\omega\Gamma(P)}\left[1+\sum_{n=1}C_n(\ln\omega)^n\right], \tag{11}$$

where

$$P=4G\ln t/t',\quad C_1=-\tfrac{1}{2},\quad C_2=(3P+14)/24(P+1),\quad C_3=-(P^2+14P+24)/48(P+1)(P+2),\ \text{etc.} \tag{12}$$

The series in Eq. (11) is rapidly convergent (C_3 for the calculation below is equal to -0.2) and will be of little importance as long as ω is small (compared to $\omega\approx 5$).

An important feature of Eq. (8) is that in order to determine $F_2(\omega,t)$ one need only know the structure function for ω' less than ω at some given t'. This has two advantages. First, the region of small ω is most accessible experimentally for large q^2. Second, as long as ω is sufficiently small, there will be a strong enhancement of the region $\omega'\approx 1$ in Eq. (8) due to the rapid falloff of the structure function near threshold. This allows one to construct an excellent approximation to Eq. (8) for $1\leq\omega\leq 4$.

If the structure function vanishes like a power d of $\omega-1$ as $\omega\to 1$, then it follows from Eq. (8) that

$$\frac{F_2(\omega,t)}{F_2(\omega,t')}\underset{\omega\to 1}{\approx}\left(\frac{t}{t'}\right)^{0.69G}\frac{(\ln\omega)^P\Gamma(d+1)}{\Gamma(d+1+P)}[1+0(\omega-1)].$$

Experimentally, one has from the energy range of the Stanford Linear Accelerator Center (SLAC) that $d=3$,[9] so that

$$R(\omega;q^2,q'^2)=\frac{F_2(\omega,t)}{F_2(\omega,t')}\cong\left(\frac{t}{t'}\right)^{0.69G}\frac{6(\ln\omega)^P}{\Gamma(4+P)} \tag{13}$$

should be an excellent representation of R for ω near threshold.

In order to get an idea of how large the deviations from scaling might be, as well as their ω dependence, let us examine Eq. (13) for the red-white-blue SU(3)′ model, where $G=\frac{4}{27}$, with the scale parameter $\mu=1$ GeV, and set $q'^2=5\ \mathrm{GeV}^2$ and $q^2=50\ \mathrm{GeV}^2$. In that case $p=0.526$. The above formula then yields

$$R(\omega;50,5)=0.54(\ln\omega)^{0.526}. \tag{14}$$

Explicit numerical integration of Eq. (8), using the exact Eq. (10) for T, and the experimental data for $F_2(\omega, 5)$,[9] show that Eq. (14) is extremely accurate for $1 \leq \omega \leq 4$ (the error is 1% at $\omega = 2$, 3% at $\omega = 3$).[10]

This ratio then rises rapidly from threshold, where large deviations from scaling are expected, $R(1.1) = 0.16$, to about 0.45 at $\omega = 2$, 0.57 at $\omega = 3$, and 0.65 at $\omega = 4$. The ratio then increases slowly, approaching 1 in the vicinity of $\omega \approx 15$.[11] The actual numerical values of R are very insensitive to the number of quark triplets [as long as the strong gauge group is SU(3)′], but are, of course, more sensitive to our choice of q'^2 and μ. Thus $R(2)$ will decrease (increase) by about 10% if we let μ^2 equal $\frac{1}{2}$ (2) and compare $q^2 = 50$ GeV^2 with $q'^2 = 2.5$ GeV^2 (10 GeV^2).[10]

Most interesting is the ω dependence of the ratio R for small ω predicted by Eq. (13). This logarithmic behavior is characteristic of all asymptotically free gauge theories, being a direct consequence of the large-N behavior of the coefficients A_N. Luckily this is precisely the region which is, kinematically,[12] most accessible for large q^2. Thus a measurement of $R(\omega)$ near threshold for large q^2 would provide a strong test of asymptotic freedom.

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‡Alfred P. Sloan Foundation Fellow.

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[3]H. Georgi and H. D. Politzer, Phys. Rev. D 9, 416 (1974).

[4]D. J. Gross and F. Wilczek, Phys. Rev. D 9, 980 (1974).

[5]H. D. Politzer, to be published.

[6] The smallness of the asymptotic form of the effective coupling constant is a necessary, but not sufficient, condition for the onset of asymptotic behavior. To determine the actual approach to asymptotic behavior one would have to know how the effective coupling approaches its asymptotic form. Better yet one can use asymptotic sum rules to estimate the value of $\bar{g}^2$, as well as the point at which asymptotic behavior sets in. Of particular utility is the ratio of longitudinal to transverse structure functions, whose asymptotic size can easily be determined.

[7]This trick has been employed recently by G. Parisi [Phys. Lett. 43B, 207 (1973)] to estimate the deviations from scaling in a model in which the moments fall like powers of q^2.

[8]For $N = 0$ I am, of course, overestimating A_N (which should vanish). The formula is extremely good for $N > 0$. As far as the quantitative estimates made in the text are concerned, the effect of this approximation is negligible since I only discuss small values of ω, $1 \leq \omega \leq 4$, which are sensitive to the large-N behavior of A_N. Also one notes that the structure function actually receives contributions from three separate operators, each with its own anomalous dimensions. Again this will be irrelevant for discussing the extrapolation to large q^2 for values of ω not too large, since in this region the falloff is dictated by the A_N's quoted in the text.

[9]G. Miller *et al*. Phys. Rev. D 5, 528 (1972). We can safely use the "scaling" function determined at SLAC in order to extrapolate, since for q^2 ranging from 5 to 10 GeV^2, say, we expect from the above formula only 10% deviations from scaling even when $\omega = 2$. However the deviations increase again as we approach threshold. We expect, say, a 25% decrease of F_2 at $\omega = 1.2$ when q^2 is increased from 5 to 10 GeV^2. This raises the possibility that a precise measurement of F_2 at SLAC in the vicinity of $\omega = 1.2$ could show a breakdown of scaling.

[10]These errors are, of course, all in the noise, compared to the 25% or so theoretical uncertainty in using the asymptotic form of the moments in a region in which the expansion parameter is roughly 0.27.

[11]For larger values of ω, R will increase with increasing q^2. We also note that Regge behavior is consistent with Eq. (8). Thus if $F(\omega, q'^2)$ exhibits Regge behavior when $\omega \to \infty$, so will $F_2(\omega, q^2)$. Also if one inserts $F_2(\omega, q'^2) = 1$ into Eq. (8), one reproduces for $F_2(\omega, q^2)$ the functions constructed in Ref. 3 to illustrate the breakdown of scaling. There, however, the deviations from scaling were underestimated, since the actual threshold dependence of $F_2(\omega, q'^2)$ was not used.

[12]Although this region is favored by kinematics for large q^2 there are other experimental problems. Among these are the fact that the structure functions themselves vanish rapidly near $\omega = 1$, and that nuclear effects (if one uses a heavy target) increase in importance near threshold.