$\theta(\lambda)$ approaches a constant along the null ray, and the two modes travel independently of each other towards some distant observer.

(4) The phase associated with the two normal modes is $S+\theta$. In the WKB approximation ($\theta \ll S$) the gradient of the phase S yields the propagation vector k^m of the electromagnetic (or gravitational) wave trains. It points along the direction along which a train of waves is propagating. By contrast, the gradient of the beating phase θ,

$$(8\pi G/c^4)^{1/2} e_a \bar{e}^{ab} F_{bm}, \qquad (27)$$

is the propagation vector of the beat-frequency waves that modulate the wave train in accordance with Eq. (25). Thus along the null ray of a propagating WKB mode there is an alternate bunching of electromagnetic and gravitational energy. The separation between the events of maximum bunching is one quarter of a beating cycle.

(5) The time-averaged (over one cycle) stress energies for electromagnetic and gravitational radiation are given, respectively, by

$$T_E^{ab} = (1/8\pi)\mathfrak{A}_E^2 k^a k^b,$$

$$T_G^{ab} = (1/64\pi)(c^4/G)\mathfrak{A}_G^2 k^a k^b.$$

Over a beat cycle the ratio of these two energies is, according to Eq. (25), of order unity:

$$T_E^{ab}/T_G^{ab} = (8G/c^4)\mathfrak{A}_E^2/\mathfrak{A}_G^2 = 1.$$

In view of the above observations one expects, for example, that all electromagnetic radiation, regardless of how it is produced, will ultimately be converted totally into gravitational radiation. The only proviso is that the wave train move in a background electromagnetic field long enough for the beating phase to change by $\Delta\theta = \pi/2$ along the null ray.

Thus a charged black hole acts as a catalyst for converting suitably polarized and directed electromagnetic radiation totally into gravitational radiation. In other words, the effective coupling between gravitational radiation and moving charged matter is of the same order as that between electromagnetic radiation and charged matter.

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[2] F. Zerilli (to be published) has gone a long way toward finding exact equations for the normal modes of a Reissner-Nordstrøm black hole.

[3] M. Johnston, R. Ruffini, and F. Zerilli, Phys. Rev. Lett. 31, 1317 (1973), have found numerically that for matter moving near a Reissner-Nordstrøm black hole the generated electromagnetic radiation is of the same order of magnitude as gravitational radiation.

[4] V. Moncrief (to be published) has found the odd-parity normal modes of a Reissner-Nordstrøm black hole.

[5] R. A. Isaacson, Phys. Rev. 166, 1263, 1272 (1969).

[6] In these perturbed equations we have already assumed that the potentials on the left-hand side satisfy the Lorentz gauge. In this paper the Latin subscripts range over 0, 1, 2, 3.

[7] See, for example, R. P. Geroch, J. Math. Phys. (N.Y.) 13, 956 (1972), Appendix B.

[8] More precisely, the quasi-particle states associated with the normal modes should perhaps be called *electromagnetic gravitons* (or equivalently, *gravitational photons*).

Separability in Directly Interacting Relativistic Particle Systems*

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We present a representation of the Poincaré group corresponding to a directly interacting system of particles valid to order $1/c^2$ which satisfies the condition of separability of the interaction.

In 1953, Bakamjian and Thomas[1] implemented a program proposed by Dirac[2] in 1949 in which a relativistic dynamics of interacting particles was to be constructed using as dynamical variables only the canonical position, momentum, and spin variables for a finite number N of particles. Somewhat later one of the present authors[3] rediscovered the results of Bakamjian and Thomas by a method employing an integration of the Lie algebra for the Poincaré group using a formal expansion in powers of $1/c^2$. In that paper it was stressed that for such a theory to be physically acceptable it should possess a property called separability of the interaction. This requires that if the system of particles is divided in any way into two subsystems infinitely far removed from one another, the representation of the Poincaré group for the total system should assume the form of a direct product of the representations of the Poincaré group corresponding to each of the subsystems. While this problem of constructing separable interactions was not solved in that paper, Coester[4] later showed that one could construct a separable interaction for a theory of the Bakamjian-Thomas type in the case of a system of three particles; however, the method could not be extended to a larger number.

The present authors recently became aware of a pair of papers which appeared in 1959, the first by Shirokov[5] and the second by Zhivopistsev, Perolomov, and Shirokov[6] (ZPS), which also approached the direct relativistic interaction problem by an expansion in powers of $1/c^2$ but which did not attempt to proceed beyond the order $1/c^2$. In the first paper a solution is obtained by Shirokov for a system of only two particles yielding a Hamiltonian covariant to order $1/c^2$ and trivially satisfying separability. The extension to an N-particle system by ZPS consists in taking the relativistic interaction to terms of order $1/c^2$ to be the sum over all pairs of the two-particle interaction terms found by Shirokov. However, except in certain special instances we have been able to show that such a result cannot be both covariant and separable.

On the basis of the work of Refs. 3 and 6 we have been able to produce a general construction of a representation of the Poincaré-group generators for a system of N particles which is both covariant to order $1/c^2$ and possesses a separable interaction. If the nonrelativistic interaction between particles consists of the sum of two-body terms each of which depends only on the spatial separation of the two particles, then our result reduces to that of ZPS. What is novel about our more general solution is that in the presence of spin-dependent or momentum-dependent, or, if applicable, isospin-dependent, two-body potentials, the relativistic (order $1/c^2$) interaction must generally include three-body as well as two-body terms in order to have both separability and covariance to the indicated order. This is reminiscent of the results of relativistic field theories which also generally produce effective many-body forces when interaction is produced by a mediating field, rather than directly.[7]

Space does not permit a derivation of our result here even though the derivation is quite straightforward. However, we can quote the results for the generators of the Poincaré group, and it may then be directly verified by the reader that they satisfy the well-known commutation relations[8] for these generators while their separability will be quite manifest.

The generators of space translations (momentum), space rotations (angular momentum), time translations (Hamiltonian), and Lorentz transformations (boost operator) are written, respectively, as $\vec{P}$, $\vec{J}$, H, and $\vec{K}$. The representation of the first two are given by[1]

$$\vec{P}=\sum_\mu \vec{p}_\mu, \quad \vec{J}=\sum_\mu(\vec{r}_\mu\times\vec{p}_\mu+\vec{s}_\mu),$$

where $\vec{r}_\mu$, $\vec{p}_\mu$, and $\vec{s}_\mu$ are the respective position, momentum, and spin of the μth particle. The Hamiltonian is given by

$$H=T+U^{(0)}+U^{(1)}+\dots,$$

$$T=\sum_\mu(m_\mu c^2+p_\mu^2/2m_\mu-p_\mu^4/8m_\mu^3c^2\dots).$$

The nonrelativistic interaction $U^{(0)}$ is assumed to be given by

$$U^{(0)}=\tfrac{1}{2}\sum_{\mu,\nu}u_{\mu\nu}^{(0)},$$

where $u_{\mu\nu}^{(0)}$ is a function of $\vec{r}_{\mu\nu}\equiv\vec{r}_\mu-\vec{r}_\nu$, $\vec{p}_{\mu\nu}\equiv(m_\nu\vec{p}_\mu-m_\mu\vec{p}_\nu)/M_{\mu\nu}$ (with $M_{\mu\nu}=m_\mu+m_\nu$), $\vec{s}_\mu$, and $\vec{s}_\nu$, which is rotationally invariant, symmetric in its two subscripts, and zero if the subscripts are equal. This implies that it also commutes with $\vec{R}_{\mu\nu}=(m_\mu\vec{r}_\mu+m_\nu\vec{r}_\nu)/M_{\mu\nu}$, $\vec{P}_{\mu\nu}=\vec{p}_\mu+\vec{p}_\nu$, and with $\vec{J}_{\mu\nu}=\vec{r}_\mu\times\vec{p}_\mu+\vec{r}_\nu\times\vec{p}_\nu+\vec{s}_\mu+\vec{s}_\nu$. $U^{(0)}$ will be *separable* if each $u_{\mu\nu}^{(0)}$ vanishes sufficiently rapidly as $r_{\mu\nu}\to\infty$.[9] We shall return to $U^{(1)}$, the correction to the interaction of order $1/c^2$, shortly.

A particular solution for the boost operator to order $1/c^2$ is then given by

$$\vec{K}=\sum_\mu[m_\mu\vec{r}_\mu-t\vec{p}_\mu]+\frac{1}{2c^2}\sum_\mu\left[\left(\vec{r}_\mu\frac{p_\mu^2}{2m_\mu}+\frac{p_\mu^2}{2m_\mu}\vec{r}_\mu\right)-\frac{\vec{s}_\mu\times\vec{p}_\mu}{m_\mu}\right]+\vec{V}^{(1)},$$

where

$$\vec{V}^{(1)}=\frac{1}{2c^2}\sum_{\mu,\nu}\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)},$$

and this is also clearly separable. ($\vec{V}^{(0)}$ is discussed in Ref. 3.)

The heart of the problem is the determination of $U^{(1)}$ and here the critical commutator of the Lie algebra is $[\vec{K},H]=i\vec{P}$. It can be shown that, to within an integration constant, $U^{(1)}$ can always be determined from this commutator provided only that $U^{(0)}$ commute with $\vec{R}$, $\vec{P}$, and $\vec{J}$, where $\vec{R}$ denotes the nonrelativistic center of mass

$$\vec{R}=\sum_\mu m_\mu\vec{r}_\mu/\sum_\mu m_\mu.$$

Furthermore, with the separable $\vec{V}^{(1)}$ above, the solution for $U^{(1)}$ can be written as the sum of two terms $U_2^{(1)}$ and $U_3^{(1)}$. The first of these is the sum of two-body terms and is identical with the result of ZPS, though written here slightly differently:

$$U_2^{(1)}=\frac{1}{2c^2}\sum_{\mu,\nu}\Bigg\{-\frac{P_{\mu\nu}^2}{2M_{\mu\nu}^2}u_{\mu\nu}^{(0)}+\frac{i(m_\nu-m_\mu)}{4m_\mu m_\nu M_{\mu\nu}}[(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})p_{\mu\nu}^2+p_{\mu\nu}^2(\vec{P}_{\mu\nu}\cdot\vec{r}_{\mu\nu}),u_{\mu\nu}^{(0)}]+$$
$$+\frac{i}{4M_{\mu\nu}^2}[(\vec{r}_{\mu\nu}\cdot\vec{P}_{\mu\nu})(\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu})+(\vec{P}_{\mu\nu}\cdot\vec{p}_{\mu\nu})(\vec{P}_{\mu\nu}\cdot\vec{r}_{\mu\nu}),u_{\mu\nu}^{(0)}]$$
$$-\frac{i}{2M_{\mu\nu}}\left[\left(\frac{\vec{s}_\mu}{m_\mu}-\frac{\vec{s}_\nu}{m_\nu}\right)\times\vec{p}_{\mu\nu}\cdot\vec{P}_{\mu\nu},u_{\mu\nu}^{(0)}\right]\Bigg\},\qquad(1)$$

where the commutation properties of $u_{\mu\nu}^{(0)}$ with $\vec{R}_{\mu\nu}$, $\vec{P}_{\mu\nu}$, and $\vec{J}_{\mu\nu}$ have been explicitly used.

However, if the commutator $[\vec{V}^{(1)},U^{(0)}]$ which enters into the commutation relation for $[\vec{K},H]$ fails to vanish [and this will generally be the case if $u_{\mu\nu}^{(0)}$ depends on spin or momentum (or isospin) operators for the two particles μ and ν], then it will consist of three-body terms. To satisfy the Lie algebra of the generators it is then necessary that $U^{(1)}$ contain three-body terms, which we write in the form[10]

$$U_3^{(1)}=\frac{i}{2c^2}\sum_{\mu,\nu,\sigma}\left\{\frac{(\vec{p}_\mu+\vec{p}_\nu+\vec{p}_\sigma)}{m_\mu+m_\nu+m_\sigma}\cdot[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]+[\vec{R}_{\mu\nu}u_{\mu\nu}^{(0)},u_{\mu\sigma}^{(0)}]\cdot\frac{(\vec{p}_\mu+\vec{p}_\nu+\vec{p}_\sigma)}{m_\mu+m_\nu+m_\sigma}\right\}.\qquad(2)$$

It will be noted that this term is also separable as a consequence of the assumed properties of the $u_{\mu\nu}^{(0)}$ as $r_{\mu\nu}\to\infty$. The full result is then a covariant representation of the Lie algebra for all the generators of the Poincaré group to order $1/c^2$, with a separable interaction.

We close with a few further remarks. We note first that the above solutions for $\vec{V}^{(1)}$ and $U^{(1)}$ are *particular* solutions of the Lie algebra. A more general separable form for $\vec{V}^{(1)}$, consisting still of two-body terms only, is obtained from that given above by adding to it

$$\Delta\vec{V}^{(1)}=\frac{i}{2c^2}\sum_{\mu,\nu}[\vec{R}_{\mu\nu},o_{\mu\nu}^{(0)}],$$

where $o_{\mu\nu}^{(0)}$ is an arbitrary rotationally invariant function of $\vec{P}_{\mu\nu}$, $\vec{r}_{\mu\nu}$, $\vec{p}_{\mu\nu}$, $\vec{s}_\mu$, and $\vec{s}_\nu$, which is symmetric in μ and ν and zero if $\mu=\nu$. This generates a change in the above form for $U^{(1)}$, given by

$$\Delta U^{(1)}=\frac{i}{2c^2}\sum_{\mu,\nu}\left[\frac{p_{\mu\nu}^2}{2m_\mu m_\nu},o_{\mu\nu}^{(0)}\right]+\frac{i}{c^2}\sum_{\mu,\nu,\sigma}\left[u_{\mu\nu}^{(0)},\frac{o_{\mu\sigma}^{(0)}}{M_{\mu\sigma}}\right].\qquad(3)$$

In addition, a more general solution for $U^{(1)}$ can be obtained by adding to the above solution any term of order $1/c^2$ which is the sum of two-body terms, three-body terms, etc., provided that they are separable and commute with $\vec{R}$, $\vec{P}$, and $\vec{J}$. This requires for two-body terms that each commute with the corresponding two-body "contributions" to these operators, i.e., $\vec{R}_{\mu\nu}$, $\vec{P}_{\mu\nu}$, $\vec{J}_{\mu\nu}$. Similarly, for three-body terms, each must commute with

$$\vec{R}_{\mu\nu\sigma}=\frac{m_\mu\vec{r}_\mu+m_\nu\vec{r}_\nu+m_\sigma\vec{r}_\sigma}{m_\mu+m_\nu+m_\sigma},\qquad\vec{P}_{\mu\nu\sigma}=\vec{p}_\mu+\vec{p}_\nu+\vec{p}_\sigma,$$

etc. Terms satisfying these conditions are easy to construct. (It is possible to include in $U^{(0)}$ three-body, four-body, etc., terms, but this will then generally require the introduction of higher-particle-number terms in $U^{(1)}$ as a result of terms generated in the commutator $[\vec{V}^{(1)},U^{(0)}]$, except, as previously, when this commutator vanishes as when all terms involve only particle positions.)

All of the above expressions (or generalizations) for $\vec{V}^{(1)}$ and $\vec{U}^{(1)}$ are, of course, consistent with and included in the arbitrariness permitted by the Lie algebra of the Poincaré group. Thus, although imposing separability definitely restricts the relativistic corrections to the interaction to order $1/c^2$ beyond that required by relativistic invariance, the relativistic corrections are still not completely prescribed by the condition of separability.

One application of dynamical theories of this type has been to calculate relativistic corrections to phenomenological potentials between particles. For this purpose all two-body terms in $U^{(1)}$ which commute with the center of mass $\vec{R}$ of the whole system can be lumped together with $U^{(0)}$ in the phenomenological potential. However, terms like those in (1), (2), and (3) above do not generally commute with $\vec{R}$. While one can rewrite the latter by expressing all momenta in terms of the total momentum $\vec{P}$ of the system and relative momenta, and terms involving only the latter might in principle be incorporated in the phenomenological potential, one must be very careful to avoid forfeiting separability. Thus, if one works in the frame in which the total momentum of the system is zero, these latter terms will in general be many-body terms and will find no natural place in a phenomenological potential; if they are discarded, separability is destroyed.

While the results presented here would constitute a more reliable basis for attempting such relativistic corrections than any previously employed, it still suffers from the substantial degree of arbitrariness pointed out above and the need for further *physical* input is clearly indicated before any truly reliable calculations can be carried out.

The derivation of the results presented here were particularly stimulated by a conversation of one of the authors (L.L.F.) with Dr. F. Coester to whom we are, in consequence, deeply indebted.

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[7]H. Primakoff and T. Holstein, Phys. Rev. 55, 1218 (1939).

[8]These may be found in Ref. 3 in the present notation.

[9]More precisely the strong limit:

$$\lim_{a\to\infty} [\exp(i\vec{a}\cdot\vec{p}_{\mu\nu})u_{\mu\nu}^{(0)}\exp(-i\vec{a}\cdot\vec{p}_{\mu\nu})]=0.$$

[10]Despite the presence of $\vec{R}_{\mu\nu}$, one can easily see that $U_3^{(1)}$ is a translationally invariant function by noting that the commutator of $\vec{P}$ with $U_3^{(1)}$ results in a sum over terms antisymmetric in two indices, and so vanishes.

ERRATUM

HALL EFFECT OF SILVER IONS IN $RbAg_4I_5$ SINGLE CRYSTALS. T. Kaneda and E. Mizuki [Phys. Rev. Lett. 29, 937 (1972)].

The conductivity in $RbAg_4I_5$ in the dark as a function of reciprocal temperature is not straight as shown in the original Fig. 1, but somewhat exponential as is presented in the figure herewith.