Beat Frequency Oscillations near Charged Black Holes and Other Electrovacuum Geometries

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Near a charged black hole (or any other space-time permeated by a steady electromagnetic field) an electromagnetic mode oscillates in sympathy with a suitably polarized gravitational mode (and conversely). Two co-traveling modes beat against each other, exchanging totally and periodically their common energy along their history near the black hole. The black hole serves therefore as a catalyst for converting electromagnetic into gravitational radiation. Thus the effective coupling of charged matter with gravitational radiation is of the same order of magnitude as with electromagnetic radiation.

Within an astrophysical context, one cannot exclude the possibility that a charged black hole is the result of the gravitational collapse of a suitably rotating star with a nonzero net charge.[1] In order to determine the electromagnetic and gravitational radiation of such a system during or after collapse, it is very helpful if one determines the normal modes of the perturbations of a charged black hole.[2-4] We consider perturbations on a space-time permeated by an as yet unspecified electromagnetic field. In order to learn as rapidly as possible the general behavior of these perturbations, we consider them in the high-frequency (WKB) approximation.[5] Thus we focus our attention on the linearized Einstein-Maxwell system,[6]

$$\nabla^m\nabla_m\psi_{ab} = -(16\pi G/c^4)\delta T_{ab}, \tag{1a}$$

$$\nabla^m\nabla_m\psi_a = g^{mn}\delta\Gamma_{mn}{}^p F_{pa} + g^{mn}\delta\Gamma_{ma}{}^p F_{np}. \tag{1b}$$

Here g_{ab} and F_{ab} are the coefficients of the background space-time geometry and the background electromagnetic field, respectively. Perturbations in the metric (h_{ab}) and in the electromagnetic field (δF_{ab}) are obtained from the "gravitational potentials," ψ_{ab}, and the electromagnetic vector potential, ψ_a, by

$$h_{ab} = \psi_{ab} - \tfrac{1}{2}g_{ab}\psi_m{}^m, \tag{2}$$

$$\delta F_{ab} = \nabla_a\psi_b - \nabla_b\psi_a. \tag{3}$$

The interaction between the gravitational and the electromagnetic modes arises from perturbations of the electromagnetic stress energy tensor T_{ab},

$$4\pi\delta T_{ab} = \delta F_{am}F_b{}^m + \delta F_{bm}F_a{}^m - \tfrac{1}{2}g_{ab}\delta F_{mn}F^{mn} + h^{mn}F_{am}F_{bn} - \tfrac{1}{2}g_{ab}h^{mn}F_{mp}F_n{}^p - \tfrac{1}{4}h_{ab}F_{mn}F^{mn}, \tag{4}$$

and of the perturbed connection,[7]

$$\delta\Gamma_{ab}{}^p = \tfrac{1}{2}(\nabla_a h_b{}^p + \nabla_b h_a{}^p - \nabla^p h_{ab}). \tag{5}$$

In the WKB approximation, we introduce[3]

$$\psi_a = e_a\psi_E, \tag{6a}$$

$$\psi_{ab} = e_{ab}\psi_G. \tag{6b}$$

Here e_a and e_{ab} characterize the polarization for electromagnetic and gravitational radiation in the usual way,[5] and

$$\psi_E = \mathfrak{A}_E e^{iS}, \tag{7a}$$

$$\psi_G = \mathfrak{A}_G e^{iS} \tag{7b}$$

are the WKB wave functions consisting of the real slowly varying amplitudes $\mathfrak{A}_E$ and $\mathfrak{A}_G$ and the rapidly varying phase factor e^{iS}, whose gradient yields the propagation vector

$$k_a = \nabla_a S. \tag{8}$$

The traceless Lorentz gauge in the WKB approximation demands that

$$0 = \nabla^m\psi_m = ik^m e_m\psi_E, \tag{9a}$$

$$0 = \nabla^m\psi_{mb} = ik^m e_{mb}\psi_G, \tag{9b}$$

$$0 = \psi_m{}^m = e_m{}^m\psi_G. \tag{10}$$

With the help of Eqs. (2) and (3), we substitute Eqs. (4) and (5) into Eqs. (1a) and (1b), respectively, introduce the two WKB wave functions in Eqs. (7a) and (7b), contract Eqs. (1a) and (1b) against the complex conjugates $\bar{e}_{ab}$ and $\bar{e}_a$ of the polarizations in Eqs. (6a) and (6b), make use of

$$\bar{e}^m e_m = \bar{e}^{mn}e_{mn} = 1,$$

use Eqs. (9)–(10), and thus finally obtain, with the help of Eqs. (8)–(10), two coupled equations for the electromagnetic and the gravitational modes:

$$\nabla^m \nabla_m \psi_G = 2i(8G/c^4)\alpha_m k^m \psi_E, \quad (11a)$$

$$\nabla^m \nabla_m \psi_E = -2i\alpha_m k^m \psi_G. \quad (11b)$$

Here

$$\alpha_m = e_a \bar{e}^{ab} F_{bm} \quad (12)$$

provides the polarization-dependent coupling along the path of the wave vector k^m. To solve the equations, we symmetrize them by introducing the dimensionless variables

$$\Phi_G = \psi_G, \quad (13a)$$

$$\Phi_E = (8G/c^4)^{1/2}\psi_E. \quad (13b)$$

In terms of these variables the coupled equations are

$$\nabla^m \nabla_m \Phi_G = 2i(8G/c^4)^{1/2}\alpha_m k^m \Phi_E, \quad (14a)$$

$$\nabla^m \nabla_m \Phi_E = -2i(8G/c^4)^{1/2}\alpha_m k^m \Phi_G. \quad (14b)$$

Their normal modes,

$$\Phi_\pm = \Phi_G \pm i\Phi_E, \quad (15)$$

satisfy the uncoupled equations

$$\nabla^m \nabla_m \Phi_\pm = \pm 2(8G/c^4)^{1/2}\alpha_m k^m \Phi_\pm. \quad (16)$$

Each mode constitutes a linear combination of a gravitational and of an electromagnetic WKB mode,

$$\Phi_\pm = [\mathcal{A}_G \pm i(8G/c^4)^{1/2}\mathcal{A}_E]e^{iS} = \mathcal{B}_\pm e^{iS}. \quad (17)$$

According to Eqs. (14) the scalar amplitudes of the photon and graviton fields satisfy

$$\nabla^m(\mathcal{A}_G{}^2 k_m) = (16G/c^4)\alpha_m k^m \mathcal{A}_G \mathcal{A}_E, \quad (18a)$$

$$\nabla^m(\mathcal{A}_E{}^2 k_m) = -2\alpha_m k^m \mathcal{A}_G \mathcal{A}_E. \quad (18b)$$

Consequently, neither graviton number nor photon number is conserved. Nevertheless, from Eqs. (16) we discover in the usual way that the magnitude of the scalar amplitude of a normal mode, Eq. (17), satisfies

$$\nabla^m(|\mathcal{B}_\pm|^2 k_m) = 0, \quad (19)$$

where

$$|\mathcal{B}_\pm|^2 = \mathcal{A}_G{}^2 + (8G/c^4)\mathcal{A}_E{}^2. \quad (20)$$

In other words the number of *electrogravitons*[8] characterized by Eq. (17) is conserved.

If photon and graviton number are not conserved along the rays of a traveling WKB wave, how do they vary? A standard WKB analysis[3] of Eq. (16) in terms of the complex amplitude $\mathcal{B}$ of Eq. (17) reveals that

$$\nabla_m(\mathcal{B}_\pm{}^2 k^m) = \mp 2i(8G/c^4)^{1/2}\alpha_m k^m \mathcal{B}_\pm{}^2. \quad (21)$$

Introduce the phase of the *beating* between electromagnetic and gravitational WKB modes [Eqs. (7)]:

$$\theta(\lambda) = -(8G/c^4)^{1/2}\int \alpha_m(dx^m/d\lambda)\,d\lambda. \quad (22)$$

Here the line integral is taken along the null ray of the WKB mode in question. Use the integrating factor $e^{\mp 2i\theta}$; then, together with Eq. (19), Eq. (21) becomes

$$k^m \nabla_m(\mathcal{B}_\pm{}^2 |\mathcal{B}_\pm|^{-2} e^{\mp 2i\theta}) = 0. \quad (23)$$

In other words, along the direction of propagation k^m, the complex amplitudes of the two electrograviton modes obey the equation

$$\mathcal{B}_\pm = |\mathcal{B}_\pm| e^{\pm i\theta}. \quad (24)$$

Equating real and imaginary parts of the complex amplitudes $\mathcal{B}_\pm$ in Eqs. (17) and (24) yields the amplitudes of the gravitational and the electromagnetic modes, respectively:

$$\mathcal{A}_G = [\mathcal{A}_G{}^2 + (8G/c^4)\mathcal{A}_E{}^2]^{1/2}\cos\theta(\lambda), \quad (25a)$$

$$\mathcal{A}_E = (c^4/8G)^{1/2}[\mathcal{A}_G{}^2 + (8G/c^4)\mathcal{A}_E{}^2]^{1/2}\sin\theta(\lambda). \quad (25b)$$

Here the factor in square brackets is the magnitude of the conserved current of electrogravitons and is the phase angle in Eq. (22):

$$\theta(\lambda) = -\int (8G/c^4) e_a \bar{e}^{ab} F_{bm}(dx^m/d\lambda)\,d\lambda. \quad (26)$$

Besides the conclusions relating to gravitons, photons, and electrogravitons in Eqs. (18)–(20), one can make the following observations on the basis of Eqs. (17) and (25).

(1) As one follows the space-time orbit of a WKB traveling mode, one will observe that electromagnetic modes always oscillate in sympathy with gravitational modes (and vice versa) near a charged black hole.

(2) Both modes beat against each other as they propagate through space-time. The phase which characterizes the beat oscillations changes along the propagation null ray according to Eq. (26). The evolution of this phase depends upon (a) the polarization of the gravitation wave, (b) the polarization of the electromagnetic wave, (c) the ambient background electromagnetic field, and (d) the direction of the polarizations relative to the background electromagnetic field.

(3) As the propagating mode travels away from the black hole, the background electromagnetic field goes to zero, the beating ceases, the phase

$\theta(\lambda)$ approaches a constant along the null ray, and the two modes travel independently of each other towards some distant observer.

(4) The phase associated with the two normal modes is $S+\theta$. In the WKB approximation ($\theta \ll S$) the gradient of the phase S yields the propagation vector k^m of the electromagnetic (or gravitational) wave trains. It points along the direction along which a train of waves is propagating. By contrast, the gradient of the beating phase θ,

$$(8\pi G/c^4)^{1/2} e_a \bar{e}^{ab} F_{bm}, \qquad (27)$$

is the propagation vector of the beat-frequency waves that modulate the wave train in accordance with Eq. (25). Thus along the null ray of a propagating WKB mode there is an alternate bunching of electromagnetic and gravitational energy. The separation between the events of maximum bunching is one quarter of a beating cycle.

(5) The time-averaged (over one cycle) stress energies for electromagnetic and gravitational radiation are given, respectively, by

$$T_E^{ab} = (1/8\pi)\mathfrak{A}_E^2 k^a k^b,$$

$$T_G^{ab} = (1/64\pi)(c^4/G)\mathfrak{A}_G^2 k^a k^b.$$

Over a beat cycle the ratio of these two energies is, according to Eq. (25), of order unity:

$$T_E^{ab}/T_G^{ab} = (8G/c^4)\mathfrak{A}_E^2/\mathfrak{A}_G^2 = 1.$$

In view of the above observations one expects, for example, that all electromagnetic radiation, regardless of how it is produced, will ultimately be converted totally into gravitational radiation. The only proviso is that the wave train move in a background electromagnetic field long enough for the beating phase to change by $\Delta\theta = \pi/2$ along the null ray.

Thus a charged black hole acts as a catalyst for converting suitably polarized and directed electromagnetic radiation totally into gravitational radiation. In other words, the effective coupling between gravitational radiation and moving charged matter is of the same order as that between electromagnetic radiation and charged matter.

[1]R. Ruffini and A. Treves, Astrophys. Lett. 13, 109 (1973).

[2]F. Zerilli (to be published) has gone a long way toward finding exact equations for the normal modes of a Reissner-Nordstrøm black hole.

[3]M. Johnston, R. Ruffini, and F. Zerilli, Phys. Rev. Lett. 31, 1317 (1973), have found numerically that for matter moving near a Reissner-Nordstrøm black hole the generated electromagnetic radiation is of the same order of magnitude as gravitational radiation.

[4]V. Moncrief (to be published) has found the odd-parity normal modes of a Reissner-Nordstrøm black hole.

[5]R. A. Isaacson, Phys. Rev. 166, 1263, 1272 (1969).

[6]In these perturbed equations we have already assumed that the potentials on the left-hand side satisfy the Lorentz gauge. In this paper the Latin subscripts range over 0, 1, 2, 3.

[7]See, for example, R. P. Geroch, J. Math. Phys. (N.Y.) 13, 956 (1972), Appendix B.

[8]More precisely, the quasi-particle states associated with the normal modes should perhaps be called *electromagnetic gravitons* (or equivalently, *gravitational photons*).

Separability in Directly Interacting Relativistic Particle Systems*

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We present a representation of the Poincaré group corresponding to a directly interacting system of particles valid to order $1/c^2$ which satisfies the condition of separability of the interaction.

In 1953, Bakamjian and Thomas[1] implemented a program proposed by Dirac[2] in 1949 in which a relativistic dynamics of interacting particles was to be constructed using as dynamical variables