PHYSICAL REVIEW
LETTERS

VOLUME 32 7 JANUARY 1974 NUMBER 1

Efficient Calculation of Correlation Functions for a Fokker-Planck System*

Wallace Arden Smith
Department of Physics, City College of the City University of New York, New York, New York 10031
(Received 1 October 1973)

A method is described for the efficient calculation of temporal correlations in systems whose equation of motion is a Fokker-Planck equation. The speed and accuracy of this approach is illustrated by a calculation of the intensity fluctuation and amplitude-phase correlation functions for a single-mode laser near threshold.

Interest in cooperative phenomena in systems far from thermal equilibrium has risen remarkably in recent years. Fokker-Planck equations have been an essential theoretical tool in analyzing these phenomena.[1] For Fokker-Planck systems the methods of calculating the dynamic properties—usually correlation functions—fall into two catagories: (1) a linearization of the equation of motion about equilibrium; and (2) a complete Green's-function solution to the equation of motion. Linearization is simple and fast when applicable. If one cannot linearize, or one needs a more accurate evaluation, the Green's-function approach provides, in principle, the desired solution. Unfortunately, in practice, the Green's-function method (requiring the solution of partial differential equations) is intractable unless the problem possesses symmetries which reduce the number of degrees of freedom. Even in problems with only one degree of freedom (requiring solutions to only ordinary differential equations) a sizable amount of numerical analysis must be performed to obtain the correlation functions. The method described here requires more effort than a linearization but considerably less than a Green's function analysis. It renders possible the investigation of a much larger class of Fokker-Planck systems than was heretofore feasible. We first describe the method and then apply it to a problem for which a detailed Green's-function analysis is already available for comparison. These calculations for a single-mode laser near threshold illustrate the speed and accuracy of this approach. A more extensive analysis of this example as well as the application of the technique to other systems will be presented elsewhere.

It is useful to note that this method yields a sequence of alternating upper and lower bounds to the exact correlation function and that these bounds systematically improve in accuracy to give a solution of any desired degree of precision.

The physical systems considered here are described in terms of a probability density $P(x_i)$ in the phase space of the macroscopic physical parameters x_i. Such a statistical system is termed a Fokker-Planck system if the temporal evolution of its probability density is given by a random diffusion of the ensemble's phase points superimposed on a systematic fluidlike flow of the phase points through the phase space. In particular, we restrict our attention to systems which obey detailed balance (that is, the equilibrium probability of a phase point going from region A in phase space to region B in a time inter-

val is equal to the probability of a transition from region B to A in the same time interval) and whose drift vectors are irreversible under time reversal (that is the ith component of the velocity of the fluidlike flow is odd or even under time reversal according to whether the coordinate x_i itself is odd or even). These are physically reasonable conditions that are satisfied by many systems both in equilibrium statistical mechanics (for example, Brownian motion and noise in electrical circuits) and nonequilibrium statistical mechanics (for example, threshold behavior of laser and parametric oscillators and the convective instability of fluids).[1]

Consider systems described by a Fokker-Planck equation,

$$\frac{\partial P}{\partial t} = -LP = -\frac{\partial}{\partial x_i} A_i P + \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} P, \tag{1}$$

which obeys detailed balance and whose drift vectors are irreversible under time reversal.[2] The static equilibrium properties of such a system are readily calculated from the equilibrium density $P_0(x)$. The dynamic properties can be expressed most conveniently in terms of the Hermitian differential operator $H = P_0^{-1/2} L P_0^{1/2}$.[2] In particular the autocorrelation function of some quantity $f[x(t)]$ can be written as

$$\langle f[x(t)]\, f[x(0)]\rangle = \langle\langle P_0^{1/2}(x) f[x]\, e^{-Ht} f[x]\, P_0^{1/2}(x)\rangle\rangle, \tag{2}$$

where H acts to the right and the double angular brackets denote integration over the whole phase space. The usual Green's-function approach is to expand H in terms of its eigenvalues and eigenfunctions, and evaluate Eq. (2) as an infinite sum.

In this note two facts are exploited to calculate the correlation function. First, its initial derivatives,[3] $\mu_n = \langle\langle P_0^{1/2}(x) f[x] (-H)^n f[x]\, P_0^{1/2}(x)\rangle\rangle$, are readily evaluated. Second, since the eigenvalues of H are real and non-negative, these initial derivatives form a Stieljes sequence.[4] The correlation function can then be approximated as a finite sum,[5]

$$S_N(t) = \sum_i^N a_n e^{-\lambda_n t}, \tag{3}$$

whose parameters, a_n and λ_n, are determined by equating the approximation's initial derivatives to those calculated for the correlation function. [Given an even number $2N$ of initial derivatives, one determines N a_n's and N λ_n's, while for an odd number $2N-1$ of initial derivatives, $\lambda_1 = 0$, and one determines N a_n's and $N-1$ λ_n's.] Moreover, as more initial derivatives are used, the approximations form a monotonic decreasing sequence of upper bounds (for an odd number of initial derivatives) and a monotonic increasing sequence of lower bounds (for an even number of initial derivatives).[5] Let us emphasize again that these approximants are *rigorous bounds* to the exact correlation function.

There are a variety of ways to calculate the a_n's and λ_n's.[5] A particularly simple one stems from the fact that the approximation (3) satisfies an Nth order differential equation with constant coefficients,

$$(\sum_0^N c_m d^m/dt^m) S_N(t) = 0. \tag{4}$$

This equation and its first $N-1$ derivatives evaluated at $t=0$ provide a set of linear equations (involving μ_0 through μ_{2N-1}) the the constants c_m. $S_N(t)$ is then just the solution to this equation with the given initial conditions.[6]

To evaluate the efficacy of this approach, consider the fluctuations in a tuned single-mode laser near threshold. The description of this system in terms of a Fokker-Planck equation is well established. Moreover, since an extensive Green's-function analysis has been performed for this system,[7] detailed results are available for comparison. The equation of motion for the probability density $P(u,u^*)$ of the slowly varying complex electric field amplitude $u(t) = I(t)^{1/2} e^{i\varphi(t)}$ is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial u}[(p - I) u P] - \frac{\partial}{\partial u^*}[(p - I) u^* P] + \frac{\partial^2}{\partial u \partial u^*} 4P \tag{5}$$

where t, u, and $I = uu^*$ are scaled as usual and p is the pump parameter, which is positive above and negative below threshold.

The two quantities of physical interest are as follows:

(1) The amplitude-phase correlation function $\langle u^*(t) u(0)\rangle$, whose correlation time $T_u = \int_0^\infty \langle u^*(t) \times u(0)\rangle\, dt / \langle I\rangle$ gives the inverse linewidth of the laser light. Since T_u is proportional to the laser intensity well above and below threshold, the linewidth factor $\alpha(p) = \langle I\rangle / T_u$ (which varies from 2 below to 1 above threshold) is customarily used to display the threshold behavior of T_u. This cor-

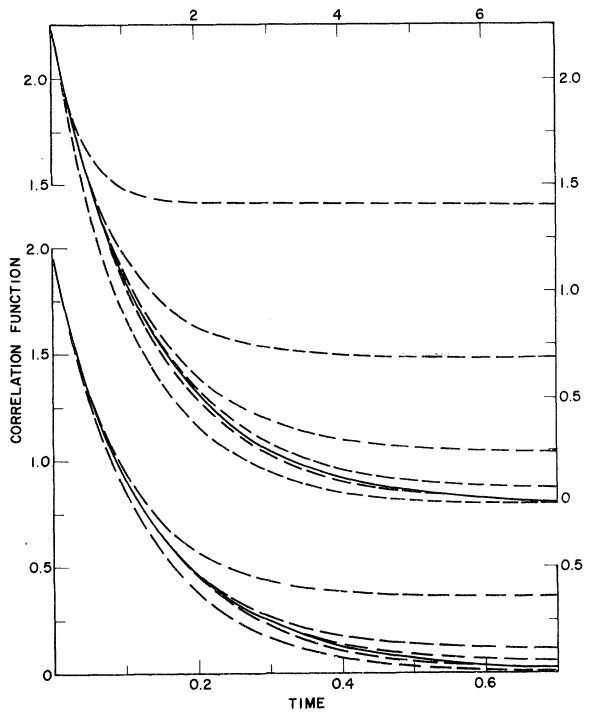

FIG. 1. Amplitude-phase correlation function for $p=2$ (upper solid curve, top and right scales), intensity fluctuation correlation function for $p=4$ (lower solid curve, bottom and left scales) and their approximants (dashed lines where distinct from solid lines) which are increasingly more precise upper (odd approximants) and lower (even approximants) bounds. Higher-order approximants are indistinguishable from the exact results.

relation function for $p=2$ and the linewidth factor as a function of p are displayed in Figs. 1 and 2

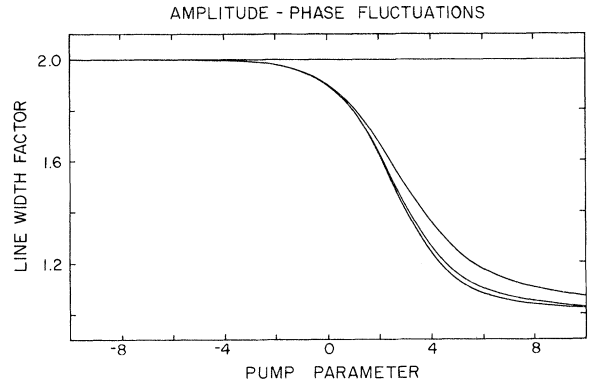

FIG. 2. Linewidth factor (lowest curve) and its first three even approximants which are increasingly more stringent upper bounds. The higher even approximants are indistinguishable from the exact result. The odd approximants give the trivial lower bound zero.

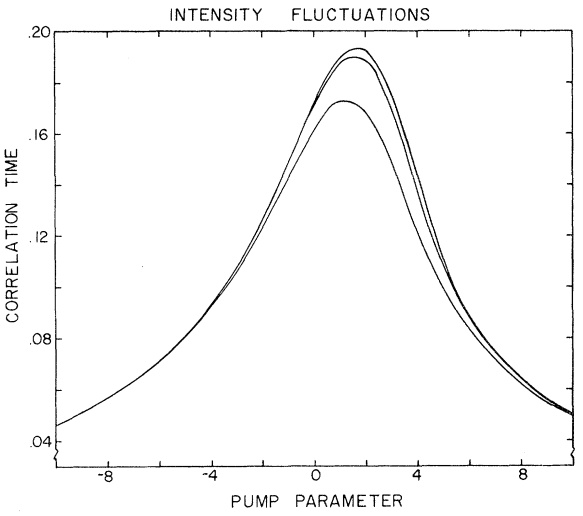

FIG. 3. Intensity-fluctuation correlation time (highest curve) and its first two even approximants which are increasingly more stringent lower bounds. The higher even approximants are indistinguishable from the exact result. The odd approximants give the trivial upper bound infinity.

along with their approximants.

(2) The intensity fluctuation correlation function $\langle \Delta I(t) \Delta I(0) \rangle$ [where $\Delta I(t) = I(t) - \langle I \rangle$], whose correlation time $T_I = \int_0^\infty \langle \Delta I(t) \Delta I(0) \rangle \, dt / \langle \Delta I^2 \rangle$ is readily measured in photon-counting experiments. This correlation function for $p=4$ and the correlation time as a function of p are displayed in Figs. 1 and 3 along with their approximants.

The first approximants in Figs. 2 and 3 reflect the initial slope of the corresponding correlation function. This is a reasonable approximation to the final result for the intensity fluctuations,[8] but evidently, a rather poor one in the amplitude-phase case.

The figures adequately summarize the accuracy of this technique. In addition, it is to be emphasized that the above results cost, in computer and human effort, less than 1% of the corresponding calculations with the Green's-function method.

*Work supported in part by the Office of Naval Research and the U. S. Army Research Office (Durham).

[1]See, for example, R. Graham, in *Springer Tracts in Modern Physics, Ergebnisse der exakten Naturwissenschaften,* edited by G. Höhler (Springer, Berlin, 1973), Vol. 66; also articles in *Synergetics,* edited by H. Haken (Teubner, Stuttgart, 1973).

[2]H. Risken, Z. Phys. 251, 231 (1972).

[3]For economy of expression we refer to the initial value as the zeroth derivative.

[4]A sequence of numbers μ_n form a Stieljes sequence if they can be written as the moments of some probability density $\rho(x)$, that is, $\mu_n = \int_0^\infty x^n \rho(x)\,dx$. For Fokker-Planck systems, $\rho(x)$ can be expressed in terms of the eigenvalues h_n and eigenfunctions Y_n of H as $\rho(x) = \sum_n |\langle\langle Y_0 f Y_n \rangle\rangle|^2 \delta(x - h_n)$.

[5]R. G. Gordon, J. Math. Phys. (N.Y.) 9, 655 (1968), gives a very readable exposition of these points and numerous citations to the literature; *The Padé Approximant in Theoretical Physics*, edited by G. A. Baker and J. L. Gammel (Academic, New York, 1970); J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Mathematical Surveys, No. 1 (American Mathematical Society, Providence, R. I., 1950). In the usual discussions the mathematical quantity considered corresponds to the Laplace transform of the correlation function. This Laplace transform admits an expansion as a continued fraction whose approximants are certain rational functions. Peeling off the Laplace transformation from these arguments yields the assertions made here.

[6]This is the procedure given an even number of initial derivatives; for an odd number the procedure is similar except that $c_0 = 0$.

[7]R. D. Hempstead and M. Lax, Phys. Rev. 161, 350 (1967); M. Lax and M. Zwanziger, Phys. Rev. A 7, 750 (1973); H. Risken and H. D. Vollmer, Z. Phys. 201, 323 (1967); H. Risken, Fortschr, Phys. 16, 261 (1968); C. D. Cantrell and W. A. Smith, unpublished.

[8]E. Jakeman and E. R. Pike, J. Phys. A: Proc. Phys. Soc., London 4, L56 (1971), have called attention to this fact.

Spherical Solitons*

Stephen Maxon and James Viecelli
University of California, Lawrence Livermore Laboratory, Livermore, California 94550
(Received 29 October 1973)

We derive a modified Korteweg–de Vries equation appropriate to small-amplitude, spherically symmetric waves. A numerical solution is obtained which differs qualitatively from the one-dimensional soliton solution.

Soliton solutions[1] in one dimension are now well known for acoustic waves propagating in a collisionless plasma of warm electrons and cold ions.[2] Experimental observation of this phenomenon is well founded.[3] The two-component (electrons and ions) fluid equations, together with Poisson's equation, reduce to the Korteweg–de Vries equation[4] in the small-amplitude approximation. The solution is a symmetric pulse moving with constant velocity, for which the square root of the peak amplitude multiplied by the width takes on a characteristic value.

In this Letter, we report results recently obtained by working with the three-dimensional, spherically symmetric version of this problem. We follow the procedure used in Ref. 2. The system of equations describing the motion is

$$\partial n/\partial r = -(e/kT)En, \tag{1}$$

$$\partial V/\partial t + V\,\partial V/\partial r = (Ze/M)E, \tag{2}$$

$$r^{-2}(\partial/\partial r)(r^2 E) = 4\pi e(ZN - n), \tag{3}$$

$$\partial N/\partial t + r^{-2}(\partial/\partial r)(r^2 NV) = 0. \tag{4}$$

N is the ion density, Ze the ion charge, M the ion mass, n the electron density, T the electron temperature, E the electric field (radial), V the ion fluid velocity, r the radial distance, and t the time. A stationary, isothermal electron fluid has been assumed.

We investigate ingoing solutions of Eqs. (1)–(4) in the small-amplitude approximation. The dispersion relation for acoustic waves in the linear approximation for long wavelengths leads us to define new dimensionless coordinates

$$\xi = -\sqrt{\epsilon}\,(r/\lambda_D + \omega_i t), \tag{5}$$

$$\eta = \epsilon^{3/2}\omega_i t, \tag{6}$$

where ϵ is the expansion parameter, λ_D the Debye length, and ω_i the ion plasma frequency. We transform Eqs. (1)–(4) from the coordinates (r,t) to the (ξ,η). Then we expand in powers of ϵ:

$$n = n_0 + n', \tag{7}$$

$$N = (1/Z)(n_0 + N'), \tag{8}$$

$$n' = \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \ldots, \tag{9}$$

$$N' = \epsilon N^{(1)} + \epsilon^2 N^{(2)} + \ldots, \tag{10}$$

$$V = \epsilon V^{(1)} + \epsilon^2 V^{(2)} + \ldots, \tag{11}$$

$$\tilde{E} = \epsilon \tilde{E}^{(1)} + \epsilon^2 \tilde{E}^{(2)} + \ldots, \tag{12}$$