lows: $\hat{h}_l^{\pm}(x) = \hat{j}_l(x) \pm i\hat{n}_l(x)$, and $\hat{h}_l^{(\pm)}(x) \rightarrow (\mp i)^{l+1} e^{\pm ix}$ as $x \rightarrow \infty$.

[5]See Calogero (Ref. 4), p. 200, where it is denoted by $\hat{D}_l^2(kr)$.

[6]We feel that this property is largely responsible for the quality of the approximation, indicating how delicate it might be to consistently invoke further corrections. In this context we note that the derivative of $u_l(r)$ found here is discontinuous at r_l. This discontinuity may be overcome by a more careful manipulation, with the result that only terms of order $V/E \ll 1$ are modified.

[7]The appropriate exact phase shifts were calculated in two independent ways by Dr. Y. Tikochinsky. We are grateful to him for his assistance.

[8]Actually this "Born" expression for δ_l is a good approximation for $l \ll L$, where phase shifts are far from being small. To demonstrate this just replace $|\hat{h}_l|^2$ in the second term of Eq. (5) by $2\hat{j}_l^2$, thus neglecting $\hat{n}_l^2 - \hat{j}_l^2$ because of its oscillatory, averaging to zero, behavior.

[9]Note that for $l \gg L$ the relevant range is a and that ka is only 1.3, not much greater than 1, for $E = 100$ MeV. It is remarkable that arguments based on oscillation still hold for such a case.

[10]This point is discussed clearly by L. L. Foldy and J. D. Walecka, Ann. Phys. (New York) 54, 447 (1969).

Threshold Effects of Diffractive Production*

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In a quite general Feynman-diagram model, diffractively produced inelastic states at high energies lead to a decrease in the total cross section below and above the threshold for the process. Nondiffractive channels produce the opposite effect. Asymptotically this agrees with the eikonal approach and the Gribov Reggeon calculus. Thus one can understand the broad dip in the pp total cross section at intermediate energies and, if the triple Pomeron decouples, the resultant intersecting-storage-rings rise.

The observed rise in the proton-proton total cross section at the CERN intersecting storage rings (ISR)[1] has generated considerable speculation on the possible existence of a new and unexpectedly large energy scale in strong interactions. On the theoretical side, the Mueller-Regge approach has led to simple and physical relations between production processes and elastic scattering.[2] The resulting analysis of diffractive fragmentation processes has led to the identification of positive contributions to the total cross section which increase with energy[3] and which may be of the same size as the observed rise. However, it has recently been pointed out that in the eikonal approach, the term which is linear in the triple-Pomeron coupling is negative definite for asymptotic energies.[4] This is in agreement with Gribov's Reggeon calculus.[5] It is our purpose here to give a simple nonasymptotic analysis, valid around threshold, of this result and the physics involved.

In those papers which try to relate the increase in the pp total cross section to the sharp rise in the inclusive proton cross section near $x \cong 1$, where x is the Feynman scaling variable, it is commonly assumed (and quite naturally) that an increase in the diffractive excitation cross section normally produces a rise in the total cross section and that the other contributions remain more or less the same.[6] In this paper, this threshold effect is re-examined and it is shown that despite expectations to the contrary, the effect on this total cross section of a rapidly rising inelastic cross section depends critically on whether the final state is diffractively produced or not. If the production mechanism is predominantly real, such as the case of real particle exchange, then the total cross section rises; however, if the production mechanism is diffractive, then the total cross section decreases with the onset of the inelastic process. The point is that in this case a rise in the fragmentation cross section normally produces twice as much fall in the cross section for events with no rapidity gap. This reversal will certainly have an effect on the decoupling theorems[7] as was briefly discussed in Ref. 4.

In order to illustrate this rather unexpected behavior which is nevertheless true in potential scattering,[8] let us examine a very simple two-channel model of the scattering matrix which is dominated by absorption. Unitarity of the S matrix will be guaranteed by writing it in matrix form in impact-parameter space. An angular

momentum expansion yields the same result. Consider the following S matrix:

$$S = \exp[-A(b) + i\sigma_x a(b)],$$

where σ_x is the symmetric and real off-diagonal Pauli matrix and $A(b)$ is positive. The energy dependence is suppressed. In this model, the transition (or production) scattering amplitude between the up and down states is real if $a(b)$ is real. The total, elastic, and production cross sections are given, respectively, by

$$\sigma_{tot} = 2\int d^2b\,[1 - e^{-A}\cos a],$$

$$\sigma_{el} = \int d^2b\,|1 - e^{-A}\cos a|^2,$$

$$\sigma_{inel} = \int d^2b\,|e^{-A}\sin a|^2.$$

Therefore, the absorption function $A(b)$ determines the absorption cross section and satisfies the equation

$$\sigma_{tot}(s) = \int d^2b\,[1 - e^{-2A}] + \sigma_{el}(s) + \sigma_{inel}(s).$$

However, if the transition scattering amplitude is diffractive (purely imaginary), then setting $a = i|a|$ in the S matrix leads to the relation

$$\sigma_{tot}(s) = \int d^2b\,[1 - e^{-2A}] + \sigma_{el}(s) - \sigma_{inel}(s),$$

where A must satisfy the relation $A \geq |a|$ for consistency with unitarity. Furthermore, if both A and a are small or are treated perturbatively, then to second order in these quantities, one finds the relation

$$\sigma_{tot}(s) = 2\int d^2b\,A - \sigma_{el}(s) - \sigma_{inel}(s) + \cdots.$$

The minus sign in front of the elastic cross section term is a reflection of the familiar phenomenon that leads to the celebrated minus sign for the net two-Pomeron cut. The diffractively produced inelastic cross section has the same sign —it could hardly be different.

Let us turn now to a proof of our result in a Feynman-diagram model in which the absorptive channels are explicitly treated. The classes of diagrams will be chosen so as to be consistent with the usual S-matrix philosophy with cluster decomposition as expressed, for example, by Abarbanel.[9]

We will consider scattering amplitudes which are built up of ladder graphs, window graphs, and fragmentation graphs. It is necessary to divide the asymptotic and intermediate states into channels which will be labeled by the number of rapidity gaps present in them. Only zero- and one-gap states will be considered but higher numbers can be trivially included. Their correspond-

ing Green's functions will be denoted by G_0 and G_1; the elastic two-body state will be dealt with explicitly and its propagator is G_e. The G's are collections of Feynman propagators in the direction of s. The real (or in general Hermitian) transition kernels are introduced as K_{e0}, K_{e1}, and K_{01}, and they contain no intermediate states of the G_0, G_1, or G_e type.

At this point, it is simplest to proceed directly to the equations for the scattering amplitudes that fully define the class of diagrams considered. They are

$$T_{ee} = K_{e0}G_0T_{0e} + K_{e1}G_1T_{1e},$$

$$T_{0e} = K_{0e}(1 + G_eT_{ee}) + K_{01}G_1T_{1e},$$

$$T_{1e} = K_{1e}(1 + G_eT_{ee}) + K_{10}G_0T_{0e}.$$

It is also convenient to introduce the positive definite operator

$$\Sigma_T \equiv (T_{ee}^* - T_{ee})/2i,$$

whose matrix element in the forward direction is proportional to the total cross section.

For example, the lowest order contribution to the transition amplitude T_{0e} is K_{0e}, which could be dominated by and certainly contains the simple multiperipheral production graph. The elastic scattering amplitude T_{ee} then contains the term $K_{e0}G_0K_{0e}$ which is immediately recognized as the ladder graphs, where G_0 contains all the propagators of the rungs. This model does not contain all graphs; in particular, the nonplanar graphs are not treated completely. Some nonplanar graphs can be included by appropriate choices for the kernels K, but others are not since when the discontinuity of the amplitude is taken, some multidimensional cuts are not included. However, since our final result agrees with the eikonal for asymptotic energies, and nonplanar graphs are required in the eikonal approach, our theorem would seem to have nonzero content. In the case of the exchange of two Pomerons, nonplanar graphs allow a cut through both Pomerons simultaneously which is a positive definite term. It is compensated by the fact that there are then also twice as many single-cut terms. For a clear and detailed analysis of these intermediate states, see the work of Botke.[10]

Since the major interest here is to explore the consequences of diffraction, it will be assumed that G_0 is dominated by its imaginary part. We will set $G_0 = -id_0$, where $d_0 \geq 0$ contains the mass-shell δ functions of the particles in the rungs of

the ladder. It is also true that at high energies, the elastic Green's function G_e is dominated by its imaginary part so that $G_e = -id_e$. It is not necessary to make this approximation in all cases but it simplifies the discussion.

Now let us compare two different physical cases, namely, the effect on the total cross section of diffractive and nondiffractive production of one-gap or fragmentation states.

(a) *Nondiffractive case.*—For this case, it will be assumed that $K_{01} = 0 = K_{10}$ and then to lowest order, $T_{1e} = K_{1e}$, which is real. The formal solution for T_{ee} is

$$T_{ee} = [1 - WG_e]^{-1}W,$$

where

$$W = K_{e0}G_0K_{0e} + K_{e1}G_1K_{1e}.$$

For later use, we define

$$-\mathrm{Im}W = A_0 + A_1,$$

where $A_{0,1}$ are positive definite and correspond to the above two terms. The total cross section operator is

$$\Sigma_T = [1 - W^*G_e^*]^{-1}(W^*d_eW - \mathrm{Im}W)[1 - G_eW]^{-1}.$$

Let us now explore the effects of the one-gap intermediate states in the situation when $G_0 = -id_0$, $G_e = -id_e$, and $G_1 = -ixd_1$, where x is a parameter that allows the one-gap states to be turned off.

One finds

$$\Sigma_T[d_0, d_e, xd_1] = [1 + (A_0 + xA_1)d_e]^{-1}(A_0 + xA_1).$$

The effect of varying x from zero to one is determined by integrating

$$\frac{d\Sigma_T}{dx} = [1 + (A_0 + xA_1)d_e]^{-1}A_1[1 + d_e(A_0 + A_1)]^{-1} \geq 0$$

which is a positive definite operator. Therefore, under the stated assumptions, the total cross section definitely increases as nondiffractively produced one-gap states are introduced. This is to be contrasted with the result in the next section.

(b) *Diffractive case.*—For this case, it is convenient to set $K_{1e} = 0 = K_{e1}$ and then one finds

$$T_{ee} = K_{e0}[1 - G_0U]^{-1}G_0K_{0e},$$

$$T_{1e} = K_{10}[1 - G_0U]^{-1}G_0K_{0e},$$

where

$$U = K_{0e}G_eK_{e0} + K_{01}G_1K_{10}.$$

It will now be assumed that G_0 and G_e are purely imaginary, but since threshold effects of the one-gap states are of interest, the real part of G_1 will not be neglected. Below threshold, G_1 is real and negative definite. T_{1e} is seen to be basically diffractive under these assumptions.

It is convenient to symmetrize the operators present in Σ_T, and after a slight rearrangement, one finds

$$\Sigma_T[d_0, zd_e, xd_1, y\,\mathrm{Re}G_1] = K_{e0}d_0^{1/2}(1+A)^{-1/2}(1+y^2L^2)^{-1}(1+A)^{-1/2}d_0^{1/2}K_{0e},$$

where

$$L = (1+A)^{-1/2}d_0^{1/2}K_{01}\,\mathrm{Re}G_1K_{10}d_0^{1/2}(1+A)^{-1/2}$$

and

$$A(z,x) = d_0^{1/2}(z\,K_{0e}d_eK_{e0} + xK_{01}d_1K_{10})d_0^{1/2}.$$

It is now obvious that Σ_T is a decreasing function of y and hence of terms proportional to $\mathrm{Re}G_1$. One can then proceed to carry out the same argument with respect to x and then z. The result is that

$$\Sigma_T[d_0, d_e, d_1, \mathrm{Re}G_1] \leq \Sigma_T[d_0, d_e, 0, 0] \leq \Sigma_T[d_0, 0, 0, 0],$$

where

$$\Sigma_T[d_0, 0, 0, 0] = K_{e0}d_0K_{0e}.$$

The above operator inequality, which is our main result, must hold for any diagonal matrix element and hence the total cross section must be less than the value given by neglecting elastic and one-gap intermediate states. This latter value is usually termed the "bare Pomeron."

One readily sees that an ever increasing contribution from the one-gap states actually damps the total cross section towards zero. Note also that by virtue of the sign reversal, the real part of the scattering amplitude coming from the fragmentation states becomes positive at sufficiently large energies.[11] This is opposite to the expectation from dispersion relations and to the idea that the fragmentation states lead to a rise in the cross section as in case (a) discussed above.

Since the real part is known to be negative at intermediate energies, it should stay negative below and in the threshold region and then start to increase and turn positive (if diffractive fragmentation dominates) when the energy is far above the effective fragmentation threshold.

It therefore seems clear that a rise in the total cross section cannot come from the turning on of a diffractively produced channel. In other words, the triple-Pomeron region must contribute negatively to the total cross section. If the (presumably) diffractively produced states do indeed have a rising cross section in pp scattering in the ISR energy range, either the basic mechanism producing the no-gap or pionization states must rise by considerably more, or there are important contributions from nondiffractive final states. The mystery of the origin of any rise in the pp total and elastic cross sections deepens, but from another point of view, our results provide a natural but perhaps not totally satisfactory explanation of the broad dip[12] in the pp cross section which is starting to go away (as an inverse power of $\ln s$ if the triple Pomeron decouples) in the ISR energy range.

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[11]While it is easy to show that the real part of the scattering amplitude is not monotonic in $\mathrm{Re}G_1$, nevertheless it does carry its sign.

[12]I am indebted to T. L. Neff for this point of view (private communication).