The long-range behaviors (5) and (8) decrease slower at infinity than the first Debye term $r_{12}^{-1/2} \times \exp(-r_{12})$, in agreement with previous three-dimensional findings.[3,7,8] Nevertheless, the most interesting result is the short-range limit where the foregoing techniques show that the most diverging nodal graph in a given order n will be the ladder one. This fact allows us to extrapolate Eqs. (4) and (7) by

$$\begin{aligned}\lim_{r_{12}\to 0} w_2(r_{12}) &\simeq -\epsilon K_0(r_{12}) + (\epsilon^2/2!)K_0^2(r_{12}) - (\epsilon^3/3!)K_0^3(r_{12}) \pm \cdots + [(-\epsilon)^P/P!]k_0^P(r_{12}) + \cdots \\ &= \exp[-\epsilon K_0(r_{12})] - 1,\end{aligned} \tag{9}$$

thus providing a quite straightforward renormalization of the short-range behavior of $g_2(r)$ in accord with Eq. (6).

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Elementary Excitations in He^3-He^4 Mixtures

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The energy spectrum of elementary excitations in dilute He^3-He^4 mixtures in investigated using a Feynman type of wave function. It is found that the He^3 energy spectrum lies below the phonon-roton energy spectrum and exhibits a minimum near the roton minimum. Some consequences of this energy spectrum are discussed.

Recently, the two-roton Raman spectra of superfluid He^3-He^4 mixtures have been measured.[1,2] The roton energy Δ_4 inferred from these spectra is practically independent of He^3 concentration up to a molar concentration of 31%, the highest concentration used. The roton energy has also been obtained from measurements of the normal-fluid density in mixtures[3] by fitting with the formula $\rho_n = n_3 m_3^* + \rho_{nr} + \rho_{np}$, where n_3 is the He^3 number density, m_3^* is the He^3 effective mass, and the last two terms are the contributions of the rotons and phonons, respectively. The value of Δ_4 obtained in this way decreases markedly with concentration, e.g., $\Delta_4 \simeq 5.1°K$ at 30% molar concentration, in striking contrast to the Raman-scattering results. Pitaevski[4] suggested recently that this could be explained if the energy spectrum of He^3 excitations in mixtures exhibited a minimum similar to the roton minimum in pure He^4.

We first present a simple argument which indicates that Pitaevski is probably correct. If we construct a wave function for a localized excitation (roton) in the mixture (for example in the manner of Feynman and Cohen[5,6]) and one of the atoms in the excitation is a He^3, then we can relax the condition that the wave function be totally symmetric. Neglecting the mass difference $m_4 - m_3$, this will lead to a lowering of the energy. At small wave vectors the energy of the He^3 excitations will have the form $\hbar^2k^2/2m_3^*$ and will lie below the phonon branch. At large wave vectors this argument indicates that the energy of the He^3 excitations will probably lie below that of the rotons and exhibit a minimum. A detailed calculation given below confirms this result. This argument is not applicable to other impurities, e.g., electrons or protons in He^4, because of the very different potentials and masses.

Suppose a He^3 atom is dissolved in superfluid He^4. The ground-state wave function is $\Phi(\vec{R},\vec{r}_1, \ldots,\vec{r}_n)$, where $\vec{R}$ is the position of the He^3 and $\vec{r}_1, \ldots,\vec{r}_n$ are the positions of the He^4 atoms. The wave function for an excitation of momentum $\hbar\vec{k}$ in the liquid is taken in the form $F\Phi$, as suggested by Feynman and Cohen.[5,6] We choose

$$F = a \exp(i\vec{k}\cdot\vec{R}) + b\textstyle\sum_i f(\vec{R}-\vec{r}_i)\exp(i\vec{k}\cdot\vec{r}_i), \tag{1}$$

where a and b are coefficients to be chosen by minimizing the energy and $f(r)$ is a smooth function which vanishes for $r > d$. The effects of backflow have been neglected in (1). The energy of the excitation is

$$E_k = \frac{\hbar^2k^2}{2m_3}\frac{a^2+\alpha^{-1}b^2I_1}{a^2+2abI_2+b^2(I_1+I_3)}, \tag{2}$$

where derivatives of f have been neglected, $\alpha = m_4/m_3$, and

$$I_1 = \int d^3r\, p_{34}(\vec{r}) f^2(r), \quad I_2 = \int d^3r\, p_{34}(\vec{r}) f(r) \exp(i\vec{k}\cdot\vec{r}),$$
$$I_3 = \int d^3r_1\, d^3r_2\, p_{344}(\vec{r}_1,\vec{r}_2) f(r_1) f(r_2) \exp[i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)]. \tag{3}$$

In (3), $p_{34}(\vec{r})$ is the probability of finding a He^4 atom at $\vec{r}$ and $p_{344}(\vec{r}_1,\vec{r}_2)$ is the probability of finding a He^4 at $\vec{r}_1$ and another at $\vec{r}_2$ if, in each case, the He^3 is at the origin. Minimizing (2) with respect to a/b and choosing the lower-energy symmetric state ($a/b > 0$) gives

$$E_k = \frac{\hbar^2k^2}{2m_3}\frac{1}{1+G_k}, \quad G_k = \frac{2\alpha I_2^2}{(D^2+4\alpha I_1 I_2^2)^{1/2} - D}, \tag{4}$$

where $D = \alpha(I_1 + I_3) - I_1$. We may approximate the integrals in (3) as follows. We choose $f(r) = 1$ for $r < d$ and zero otherwise. When $d > r_0$, the interparticle spacing I_1 is a measure of the number of He^4 atoms in the excitation. This is left as a variable parameter. For $kd > 1$ a good approximation is $I_2 \simeq p_{34}(k)$. In I_3 we use the superposition approximation, $n_0 p_{344}(\vec{r}_1,\vec{r}_2) = p_{34}(\vec{r}_1)p_{34}(\vec{r}_2) \times p_{44}(\vec{r}_1-\vec{r}_2)$, where n_0 is the He^4 number density; then for $kd > 1$ a good approximation is $I_3 \simeq I_1 p_{44}(k)$. This is probably an underestimate, as we have not taken into account the most favorable configurations of the three atoms. The energy is not very sensitive to I_1 for $1 < I_1 < 8$. For I_1 very large, $G_k = \alpha[1 + p_{44}(k)] - 1$ and E_k goes over into the roton energy found by Feynman.[5] This indicates that the addition of a small amount of He^3 will not affect the roton gap Δ_4. If we allow $d \to 0$ and $I_1 \ll 1$, it is necessary to include the kinetic-energy terms arising from ∇f and the lowest energy will be obtained with $b = 0$ (a single He^3 excitation). This is appropriate for small k. To estimate the energy for $k \simeq 2$ Å^{-1}, we need to know the structure factor $p_{34}(k)$. This was obtained by putting $p_{34}(k) = p_{44}(k) - \Delta(k)$. For $p_{44}(k)$ the experimental values of Goldstein and Reekie[7] were used. $\Delta(k)$ was estimated from the calculations of Massey, Woo, and Tan[8] by taking the Fourier transform of the difference of their calculated pair correlation functions, $g_{44}(r) - g_{34}(r)$. The results are given in Fig. 1. Because of the larger zero-point motion and larger volume (by about 30%) of a He^3 atom in He^4, the maximum in $p_{34}(k)$ is at a smaller value of k, ~1.9 Å^{-1}, than

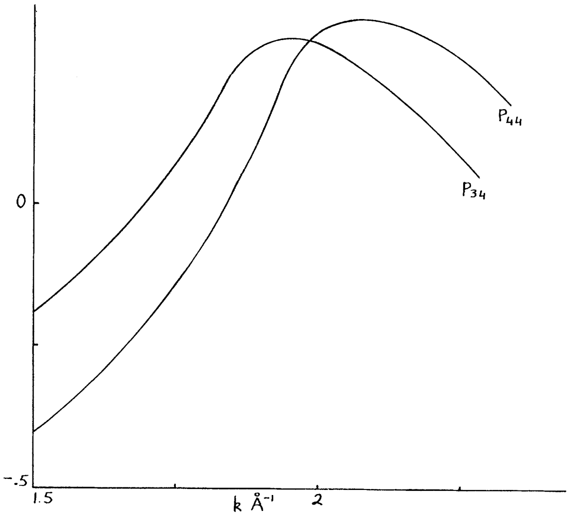

FIG. 1. The structure factors $p_{34}(k)$ and $p_{44}(k)$.

that in p_{44} and p_{34} is somewhat broader. We represent the He^3 spectrum for $k \sim 1.9$ Å^{-1} by $E_k = \Delta_3 + (\hbar^2/2\mu_3)(k - k_3)^2$, with $k_3 = 1.9$ Å^{-1}. From (4) and the data of Fig. 1 we find $G_{k_3} = 0.64$, $\Delta_3 = 16.5$°K, and $a/b = 0.4$ for $I_1 = 2$. The value of Δ_3 varies by 10% for $1 < I_1 < 8$. Δ_3 is probably too large, as we have omitted backflow effects. The roton gap Δ_4, calculated omitting backflow,[6] is 19°K and the ratio $\Delta_3/\Delta_4 = 0.86$. The important result is that the energy of the He^3 excitations lies below the roton spectrum. A sketch of the spectrum is given in Fig. 2. The effective mass $\mu_3 > \mu_4$, corresponding to the fact that $p_{34}(k)$ is broader than $p_{44}(k)$. To summarize, the energy spectrum of He^3 excitations in He^4 has the form $E_k = \hbar^2 k^2/2m_3{}^*$ at small wave vectors and $E_k = \Delta_3 + \hbar^2(k - k_3)^2/2\mu_3$ for $k \sim 1.9$ Å^{-1}. For convenience we refer to these latter excitations as He^3 rotons. As rotons are very localized excitations, normal rotons may also exist in the mixtures up to quite high concentrations.

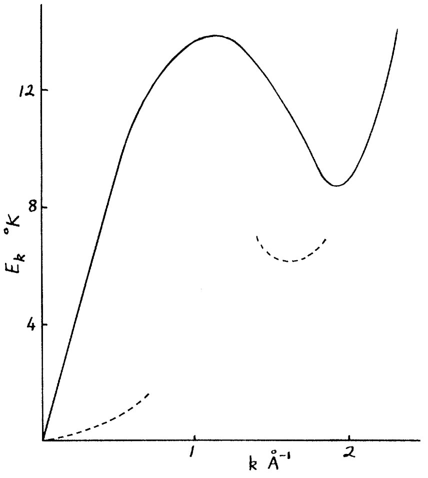

FIG. 2. Sketch of the energy spectrum of He^3 excitations (dashed line) in He^3-He^4 mixtures and the phonon-roton (solid line) energy spectrum. We have assumed Δ_3 is of the order of magnitude found by Woerner (Ref. 9).

There are a number of interesting consequences of this form for the spectrum of He^3 excitations. The impurity contribution to the normal fluid density, ρ_{ni}, the energy density E_i, and the He^3 chemical potential $\bar{\mu}_3$ are respectively given by (in the nondegenerate case)

$$\rho_{ni} = n_3 m_3{}^* \left(\frac{1 + \frac{1}{3}(\hbar^2 k_3{}^2/m_3{}^* kT) A \exp(-\beta\Delta_3)}{1 + A\exp(-\beta\Delta_3)}\right), \tag{5}$$

$$E_i = \frac{3}{2} kT \left(\frac{1 + \frac{2}{3} A(\Delta_3/kT + \frac{1}{2})\exp(-\beta\Delta_3)}{1 + A\exp(-\beta\Delta_3)}\right), \tag{6}$$

$$m_3\bar{\mu}_3 = kT\ln n_3 - \tfrac{3}{2}kT\ln(m_3{}^* kT/2\pi\hbar^2) - kT\ln[1 + A\exp(-\beta\Delta_3)], \tag{7}$$

where $\beta = (kT)^{-1}$ and $A = (2\hbar^2 k_3{}^2/m_3{}^* kT)(\mu_3/m_3{}^*)^{1/2}$. At $T = 1.5$°K we estimate $\hbar^2 k_3{}^2/m_3{}^* kT = 13$ and $A = 7$.

The measurements of ρ_n of Sobolev and Esel'son[3] have been fitted with the formula $\rho_n = \rho_{ni} + \rho_{nr}$ by Woerner.[9] At a He^3 molar concentration of 11% and $T < 1.5$°K the best fit was obtained with $\Delta_3 = 5 \pm 1.5$°K and $(\hbar^2 k_3{}^2/m_3{}^* kT)A \simeq 30$. The error in this latter value may be large (a factor of 3) and the data are not sufficiently accurate over a wide temperature range to determine A. The impurity contribution to the specific heat follows by differentiation of E_i. The He^3 chemical potential also reflects the energy spectrum of the excitations. $\bar{\mu}_3$ can be obtained from a measurement of the He^3 concentration in the vapor in equilibrium with the liquid.[10]

The He^3 dynamic structure factor $S_3(k,\omega)$ for $k \simeq k_3$ and $\omega \simeq \Delta_3$ in the nondegenerate case is

$$S_3(k,\omega) = (\pi n_3/kT)(\mu_3/m_3{}^*)^{1/2}\exp(-\beta\gamma|\omega - \Delta_3|)/[1 + A\exp(-\beta\Delta_3)], \tag{8}$$

where $\gamma = 1$, $\omega < \Delta_3$; and $\gamma = \mu_3/(m_3{}^* - \mu_3)$, $\omega > \Delta_3$. The function is broadened by the thermal distribution of the particles. In the degenerate case,

$$S_3(k,\omega) = \frac{m_3{}^*}{\pi\hbar^2}\left(\frac{2\mu_3}{\hbar^2}\right)^{1/2}\left\{(\omega - \Delta_3 + \epsilon_F)^{1/2} - \left[\left(\frac{m_3{}^*}{m_3{}^* - \mu_3}\right)(\omega - \Delta_3)\right]\right\}^{1/2}, \tag{9}$$

where ϵ_F is the Fermi energy, imaginary terms are to be omitted, and $-\epsilon_F < \omega - \Delta_3 < \epsilon_F(m_3{}^* - \mu_3/\mu_3)$. As a result of the strong absorption, it may not be possible to observe this function by neutron scattering.

The possibility also exists of observing a Raman-scattering process in which two He^3 rotons are created. The spectrum, neglecting any possible interactions between the particles, is the convolution of two factors of S_3.[11] For a frequency shift $\omega \simeq 2\Delta_3$ and $\beta|\omega - 2\Delta_3| < 1$ the spectrum is proportional to $\exp[\beta\gamma(\omega - 2\Delta_3)]$, with $\gamma = \mu_3/(m_3^* - \mu_3)$, while for $\beta(\omega - 2\Delta_3) > 1$ the spectrum varies as $(\omega - 2\Delta_3)^{-1/2}$. Again the spectrum is thermally broadened compared with the two-roton Raman spectrum. The ratio of the intensities of the two-He^3-roton and two-roton Raman spectra is approximately $(n_3/n_4)^2$, where n_3 and n_4 are the He^3 and He^4 densities, respectively.

Raman scattering in which a He^3 roton near k_3 and a He^4 roton near k_4 (where the density of states in each case is largest) are created is forbidden by momentum conservation when $k_4 > k_3$. Thus Raman scattering at a frequency shift $\omega \sim \Delta_3 + \Delta_4$ would not be expected to show any special feature.

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Critical Scattering in a Field and Below T_c

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Series expansions for spin-spin correlations of simple-cubic, bcc, and square lattice Ising ferromagnets for general field and temperature have been analyzed. To within the attainable precision (of ± 1 to ± 5%) the exponents ν' and ν^c are consistent with scaling predictions. Deviations of the scattering from Ornstein-Zernike forms are significantly larger quantitatively than above T_c; the second-moment correlation lengths verify $\xi_1^+(T)/\xi_1^-(T) = 1.96 \pm 0.03$ as $T \to T_c^\pm$.

The behavior of the spin-spin correlation functions, and thence of the critical scattering, in Ising models has been studied in considerable detail as the critical "isochore" (or $H = 0$) above T_c.[1-3] We report here the results of the first calculations by series expansion techniques which include the complete field dependence, so enabling one to study the scattering in zero field *below* T_c (i.e., along the phase boundary) and *at* the critical temperature as a function of field H. To within the comparatively low precision attainable (± 1 to ± 5%), the estimated exponent values confirm scaling predictions (but they throw little new light on "hyperscaling" relations such as $d\nu' = 2 - \alpha'$, etc.). More significantly, the calculations reveal large deviations in the form and scale of the critical scattering from the Ornstein-Zernike perdictions. These have been elucidated quantitatively and should be susceptible to experimental test (as have similar predictions above T_c).[4,5]

Series expansions were generated in powers of the variables

$$u = x^2 = \exp(-4J/k_BT) \text{ and } \quad y = \exp(-2mH/k_BT), \tag{1}$$

for the correlation functions

$$\Gamma(\vec{r}, H, T) = \langle S_{\vec{0}} S_{\vec{r}} \rangle - \langle S_{\vec{0}} \rangle \langle S_{\vec{r}} \rangle \tag{2}$$

of nearest-neighbor Ising models using the semi-invariant techniques as applied by Jasnow and