

Equilibrium Pair-Correlation Function for a Two-Dimensional Plasma

C. Deutsch

Laboratoire de Physique des Plasmas, Université Paris XI, Centre d'Orsay, 91405 Orsay, France

and

M. Lavaud

*Centre de Recherches sur les Hautes Températures, Centre National de la Recherche Scientifique,
45100 Orleans-la-Source, France*

(Received 7 August 1973)

The pair-correlation function for a one-component, two-dimensional classical plasma is investigated within the framework of the Debye approximation along the lines of the Cohen-Murphy method through a potential of average force, $w_2(r)$, up to third order in the plasma parameter $q^2/k_B T$. The $w_2(r)$ short-range behavior appears to be an easily renormalizable quantity, while the long-range behavior confirms previous three-dimensional results.

The purpose of this Letter is to investigate the next higher corrections to the Debye high-temperature approximation of the radial distribution function for a two-dimensional Coulomb system of point particles interacting via the potential $q^2 \ln(r/L)$. $L = 1$ determines the zero of the potential. The motivations for this work are numerous. The most evident one is to seek analogies or differences with the well-known three-dimensional situation. Moreover, the equilibrium properties of the two-dimensional Coulomb gas are interesting in their own right. They provide insight to the real strongly magnetized plasma problem.¹ They could also allow some insights into the condensation processes of charged particles at low temperature. In this work, we address ourselves to the well-known one-component model with a continuous background. This choice proves to be particularly well justified in the present case if one remembers² that for $k_B T > \frac{1}{2}q^2$, the one- and two-component Coulomb systems have the same equation of state, $p = (k_B T - \frac{1}{4}q^2)\rho$, with $\rho = N/V$ and $2N/V$, respectively. We consider the pair-correlation function $g_2(r)$ in the form used recently by Cohen and Murphy³ for the three-dimensional plasma:

$$g_2(r) = \exp[-w_2(r)], \quad (1)$$

where $w_2(r)$ is the potential of average force.

We analyze $w_2(r)$ in a nodal-graph expansion in the dimensionless plasma parameter $\epsilon = q^2/k_B T \ll 1$. As usual, the first-order term corresponds to the long-range resummation of the bare Coulomb potential,

$$w_2^1(r_{12}) = -2\pi \int d^2p \frac{V(p)e^{i\vec{p}\cdot\vec{r}}}{1 - \rho V(p)} = -\beta q^2 k_0 \left(\frac{r}{\lambda_D} \right), \quad \beta = (k_B T)^{-1}, \quad (2)$$

pictured in Fig. 1 and already considered by Hauge and Hemmer; here $\lambda_D^2 = k_B T / 2\pi q^2 \rho$, and $-k_B T V(p) = q^2/p^2$ denotes the Fourier transform of the Coulomb potential. The foregoing result has also been obtained by Montgomery and Vahala⁴ through the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy. It will be shown in a more detailed work⁵ that in the present case the potential-of-average-force techniques³ allow a determination order by order of the higher nodal diagrams. Therefore a graph of order n is given by its l renormalized Debye bounds and its k field points with $n = l - k$ in accord with the

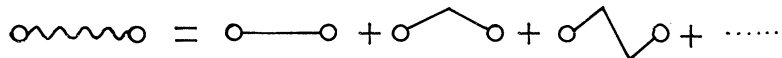


FIG. 1. First-order Debye chain.

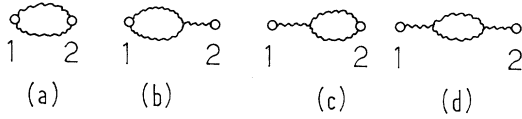


FIG. 2. Second-order graphs entering $w_2^2(r_{12})$.

Salpeter⁶ formulation of the Debye scheme. So, the second order is (see Fig. 2)

$$w_2^2(r_{12}) = (\epsilon^2/2!) \{ K_0^2(r_{12}) - 2 \int_0^{r_{12}} du u k_0^2(u) I_0(u) k_0(r_{12}) - 2 I_0(r_{12}) \int_{r_{12}}^\infty du u k_0^3(u) + \frac{1}{2} \int_0^{r_{12}} du u K_0^2(u) [-u I_1(u) K_0(r_{12}) + I_0(u) r_{12} K_1(r_{12})] + \frac{1}{2} \int_{r_{12}}^\infty du u K_0^2(u) [-K_0(u) r_{12} I_1(r_{12}) + u K_1(u) I_0(r_{12})] \}, \quad (3)$$

with r_{12} evaluated in units of λ_D . $I_n(x)$ and $K_n(x)$ refer to the modified Bessel functions of the first and second kind, respectively. Equation (3) yields in a straight-forward way the limit behaviors⁷

$$\lim_{r_{12} \rightarrow 0} w_2^2(r_{12}) \sim (\epsilon^2/2!) [\ln(\gamma r_{12}/2)]^2, \quad (4)$$

$$\lim_{r_{12} \rightarrow \infty} w_2^2(r_{12}) \sim \frac{\epsilon^2}{2!} \left(\frac{\pi}{2} \right)^{1/2} \left[\left(\frac{\pi}{2} \right)^{1/2} \frac{\exp(-2r_{12})}{r_{12}} - 0.6705 \frac{\exp(-r_{12})}{r_{12}^{1/2}} + 0.1511 r_{12}^{1/2} \exp(-r_{12}) \right], \quad (5)$$

with γ the Euler constant. Now, it is important to notice that the summability at the origin of the logarithmic potential allows us to extend to the following order in ϵ the above procedure by using the finite Fourier transforms

$$\int d^3r e^{i\vec{p} \cdot \vec{r}} (\text{Debye})^n = 2\pi \int_0^\infty dr r J_0(pr) K_0^n(r) < +\infty \quad (6)$$

for all n , while the three-dimensional analog

$$4\pi \int_0^\infty dr r \sin(pr) e^{-nr}/r^n$$

diverges for $n > 2$, so that further resummations are necessary³ in this case. Therefore we get $w_2^3(r_{12})$ as a sum of the graphs given in Fig. 3, with an explicit expression much too long to be given here and reserved for a future work.⁵ However, we are mainly interested in the limit behaviors

$$\lim_{r_{12} \rightarrow 0} w_2^3(r_{12}) = \epsilon^3 \{ - [\ln(\frac{1}{2}\gamma r_{12})]^3/3! + 0.3909 \ln(\frac{1}{2}\gamma r_{12}) \}, \quad (7)$$

$$\lim_{r_{12} \rightarrow \infty} w_2^3(r_{12}) = \epsilon^3 (\frac{1}{2}\pi)^{1/2} [-0.3073/3! + \frac{1}{2}D] r_{12}^{1/2} \exp(-r_{12}) + \text{longest chain}, \quad (8)$$

with D a finite constant and

$$\lim_{r_{12} \rightarrow \infty} [\text{longest chain}] > r_{12}^{1/2} \exp(-r_{12}).$$

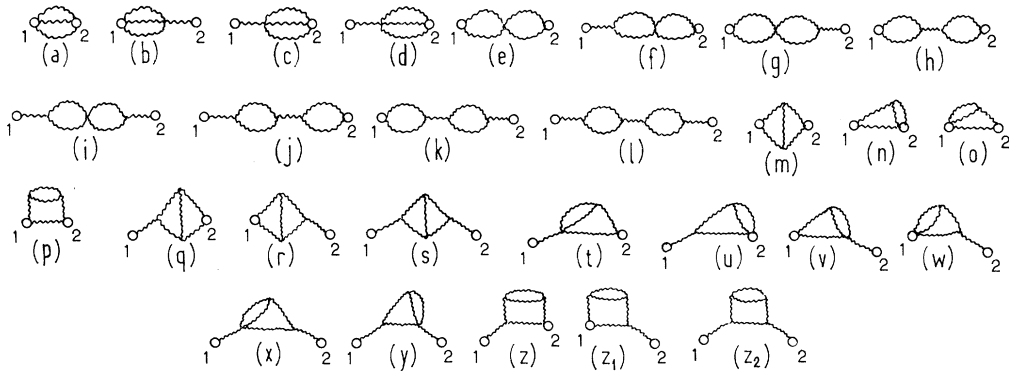


FIG. 3. Third-order graphs entering $w_2^3(r_{12})$.

The long-range behaviors (5) and (8) decrease slower at infinity than the first Debye term $r_{12}^{-1/2} \times \exp(-r_{12})$, in agreement with previous three-dimensional findings.^{3,7,8} Nevertheless, the most interesting result is the short-range limit where the foregoing techniques show that the most diverging nodal graph in a given order n will be the ladder one. This fact allows us to extrapolate Eqs. (4) and (7) by

$$\lim_{r_{12} \rightarrow 0} w_2(r_{12}) \simeq -\epsilon K_0(r_{12}) + (\epsilon^2/2!)K_0^2(r_{12}) - (\epsilon^3/3!)K_0^3(r_{12}) + \dots + [(-\epsilon)^P/P!]k_0^P(r_{12}) + \dots = \exp[-\epsilon K_0(r_{12})] - 1, \quad (9)$$

thus providing a quite straightforward renormalization of the short-range behavior of $g_2(r)$ in accord with Eq. (6).

We are indebted to Professor D. Montgomery for many enlightening discussions at Les Houches Summer School which provided the stimulation for the present work. We also thank Professor P. C. Hemmer for sending a preprint of Ref. 2 before publication and acknowledge the help of Dr. W. E. Wells in the preparation of the manuscript.

¹J. B. Taylor and B. McNamara, *Phys. Fluids* **14**, 1492 (1971).

²E. H. Hauge and P. C. Hemmer, *Phys. Norv.* **5**, 209 (1971).

³E. G. D. Cohen and T. J. Murphy, *Phys. Fluids* **12**, 1404 (1969).

⁴D. Montgomery and J. Vahala, *J. Plasma Phys.* **6**, 425 (1971).

⁵C. Deutsch and M. Lavaud, to be published.

⁶E. E. Salpeter, *Ann. Phys. (New York)* **5**, 183 (1958).

⁷We have to thank Dr. S. Klarsfeld for his help in the evaluation of a number of quadratures.

⁸H. De Witt and F. Del Rio, *Phys. Fluids* **12**, 791 (1969).

⁹D. J. Mitchell and B. W. Ninham, *Phys. Rev.* **174**, 280 (1968).

Elementary Excitations in He³-He⁴ Mixtures

Michael J. Stephen*

Physics Department, Rutgers University, New Brunswick, New Jersey 08903

and

Laurence Mittag

Center for Theoretical Studies, University of Miami, Coral Gables, Florida 33124

(Received 28 June 1973)

The energy spectrum of elementary excitations in dilute He³-He⁴ mixtures is investigated using a Feynman type of wave function. It is found that the He³ energy spectrum lies below the phonon-roton energy spectrum and exhibits a minimum near the roton minimum. Some consequences of this energy spectrum are discussed.

Recently, the two-roton Raman spectra of superfluid He³-He⁴ mixtures have been measured.^{1,2} The roton energy Δ_4 inferred from these spectra is practically independent of He³ concentration up to a molar concentration of 31%, the highest concentration used. The roton energy has also been obtained from measurements of the normal-fluid density in mixtures³ by fitting with the formula $\rho_n = n_3 m_3^* + \rho_{nr} + \rho_{np}$, where n_3 is the He³ number density, m_3^* is the He³ effective mass, and the last two terms are the contributions of the rotons and phonons, respectively. The value of Δ_4 ob-

tained in this way decreases markedly with concentration, e.g., $\Delta_4 \approx 5.1^\circ\text{K}$ at 30% molar concentration, in striking contrast to the Raman-scattering results. Pitaevski⁴ suggested recently that this could be explained if the energy spectrum of He³ excitations in mixtures exhibited a minimum similar to the roton minimum in pure He⁴.

We first present a simple argument which indicates that Pitaevski is probably correct. If we construct a wave function for a localized excitation (roton) in the mixture (for example in the manner of Feynman and Cohen^{5,6}) and one of the