same $[=(\frac{1}{3})^{3/2}2e\pi\xi^3]$ for $B \leq \frac{2}{3}H_{c2}$, with continuous transitions for larger B.

Although our calculation has been carried out in a particularly simple regime, we expect that the general behavior we predict should be observed in more general cases. Thus, further experimental work characterizing resistive transitions in thin-film superconductors should be interesting, particularly observations of when continuous transitions to the normal state are obtained as the electric field strength is increased, and when discontinuous transitions occur. Some curves showing discontinuous transitions in thin films have been obtained by Ōgushi, Takayama, and Shibuya.[10] Unfortunately, discontinuous transitions can also occur if the thermal conductivity of the substrate is too poor to remove readily the heat generated so that macroscopic hot normal regions occur, which we have not considered. Considerable care must be taken to separate the two types of effects experimentally. The thin-film regime is favorable since the total current is reduced for a constant current density. The required current density can be further reduced by working close to the reduced T_c, although fluctuations become important and smear out the transitions in a region too close to T_c.[11]

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Ultrasonic Anomalies in Simple Metals due to Backward Scattering of the Conduction Electrons

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The acoustic attenuation for a one-band free-electron metal has been derived using a generalized scattering function. If the latter has an appreciable backward lobe, the attenuation exhibits an anomalous peak when the sound wavelength is comparable to the electron mean free path. Similar effects might be observable in the alkalis as a result of umklapp scattering processes. If so, further study could enhance our understanding of the electron-phonon interaction in these metals.

The free-electron model of a metal has the merit that Fermi-surface disturbances due to applied fields can be expressed in terms of spherical harmonics. It is well known[1] that if the scattering function $W(\vec{k}, \vec{k}')$ depends only on the angle θ between $\vec{k}$ and $\vec{k}'$, then each spherical harmonic Y_{LM} relaxes towards zero with time constant τ_L given by

$$1/\tau_L = \int [1 - P_L(\cos\theta)] W(\theta) d\Omega, \quad (1)$$

where P_L is a Legendre function, $W(\theta)$ is the differential scattering probability, and $d\Omega$ is the element of solid angle. Associated with each τ_L is an effective mean free path defined by

$$l_L = v_0 \tau_L, \quad (2)$$

where v_0 is the Fermi velocity of the electrons. A special case is that of isotropic scattering, where $W(\theta)$ is independent of θ so that all the τ_L

are equal to one another.

Bhatia and Moore[2] have shown that, whereas the electrical conductivity is proportional to τ_1, the acoustic attenuation (in the local limit $ql_2 \ll 1$, where q is the magnitude of the sound propagation vector) is proportional to τ_2. Since τ_2/τ_1 depends on $W(\theta)$, the numerical magnitude of this ratio is a possible point of contact between theory and experiment. For electron-phonon scattering in potassium, in particular, much effort has gone into the experimental[3] and theoretical[4,5] determination of τ_2/τ_1. It appears that, while this ratio is quite sensitive to the relative proportion of normal and umklapp scattering events and to the particular choice of pseudopotential used in the computation of the electron-phonon matrix elements, there is at present insufficient data to enable any given pseudopotential to be favored.

In this paper we present expressions for the acoustic attenuation valid for *all* values of the ql_L, i.e., with no restrictions on locality. The attenuation, as a function of ql_2, shows structure which depends on $W(\theta)$, so that, in principle, previous investigations can be extended using acoustic methods alone. The attenuation has been evaluated for artificial models in which $W(\theta)$ is highly directional; the resultant anomalies (by comparison with the well-known Pippard result[6] for isotropic scattering) may well have a counterpart in some real metals.

In the limit $\omega\tau_L \ll 1$, where ω is the sound frequency, and subject to the conditions of charge and current neutrality in the metal, the attenuation coefficient for transverse sound waves is given by the formula[7]

$$\alpha_t = \frac{Nmv_0q}{3\rho v_{st}} \cfrac{1.3}{5/ql_2 + \cfrac{2.4}{7/ql_3 + \cfrac{3.5}{9/ql_4 + \cdots}}}, \qquad (3a)$$

and for longitudinal waves by

$$\alpha_l = \frac{Nmv_0q}{3\rho v_{sl}} \cfrac{2.2}{5/ql_2 + \cfrac{3.3}{7/ql_3 + \cfrac{4.4}{9/ql_4 + \cdots}}}, \qquad (3b)$$

where N is the number of electrons per unit volume, m is the electron mass, ρ is the density of the metal, and v_{st} and v_{sl} are the two sound velocities. The continued fractions are expressed in terms of the dimensionless parameters ql_L, which depend on the scattering mechanism through (1) and (2). If all the ql_L are replaced by ql, where $l = v_0\tau$, then Eqs. (3) become identical to the Pippard expressions.[8,9] For arbitrary $W(\theta)$ the attenuation may be evaluated with good accuracy by truncating the continued fractions at the Lth solidus, where L is at least several times larger than ql_2. Physically, this procedure is equivalent to representing the disturbed electron distribution by a finite series of spherical harmonics; it is a property of these functions[10] that such an approximation gives a best fit, in the least-squares sense, to the actual distribution. In the local limit, $ql_2 \ll 1$, the Fermi surface is adiabatically deformed by the sound wave into an ellipsoid; the radial difference between this ellipsoid and the equilibrium sphere can always be expressed as some linear combination of those harmonics having $L = 2$. Accordingly, the deformed surface continually relaxes back to equilibrium with time constant τ_2, and the attenuation depends only on this parameter. When the problem becomes nonlocal, i.e., when ql_2 exceeds unity, the distribution begins to build up a peak on the "effective zone" of the Fermi surface.[6] Electrons traveling almost normal to $\vec{q}$ stay in phase with the sound wave longer than other electrons and are displaced more. The distribution then requires more spherical harmonics to represent it, a fact which is reflected in the slower convergence rate of the continued fractions and hence in the need for the inclusion of more terms. The attenuation then has a functional dependence on those τ_L having $L \geq 2$. For arbitrary $W(\theta)$, however, the continued fractions approach the same limits when $ql_2 \gg 1$. The limiting values are $4/\pi$ for transverse waves and $\pi/2$ for longitudinal waves. In the extreme nonlocal limit, therefore, the attenuation becomes independent of the details of the scattering mechanism.

To simulate scattering with a strong forward lobe, we define

$$W(\theta) = R_1, \quad \theta < \Theta_1,$$
$$= 0, \quad \text{otherwise}, \qquad (4)$$

where R_1 and Θ_1 are constants. Substitution into (1) then gives

$$1/\tau_L = 2\pi R_1[1 - x_1 - Q_L(x_1)], \qquad (5)$$

$$(2L+1)Q_L(x) \equiv P_{L-1}(x) - P_{L+1}(x), \qquad (6)$$

and where $x_1 \equiv \cos\Theta_1$. The special case of isotropic scattering can be recovered by setting $\Theta_1 = \pi$; the τ_L then become independent of L, as expected. In the small-angle limit $\Theta_1 \to 0$, the τ_L

decrease according to the relation

$$\tau_2/\tau_L = L(L+1)/6, \tag{7}$$

which can be derived independently under the assumption that the electrons diffuse over the Fermi surface. For small but finite Θ_1 the τ_L follow (7) for small L, but ultimately level off to a constant value for L so large that the angular separation between neighboring peaks and troughs on the Y_{LM} is less than Θ_1. In this regime the scattering is "catastrophic" in that every collision randomizes an electron.

To simulate scattering with a strong backward lobe, we define

$$W(\theta) = R_2, \quad \pi - \theta < \Theta_2,$$
$$= 0, \quad \text{otherwise}, \tag{8}$$

where R_2 and Θ_2 are constants. Substitution into (1) then gives

$$1/\tau_L = 2\pi R_2[1 - x_2 - (-1)^L Q_L(x_2)], \tag{9}$$

where $x_2 = \cos\Theta_2$. Again, the special case of isotropic scattering can be recovered by setting $\Theta_2 = \pi$. For small Θ_2, however, the τ_L behave quite differently according to whether L is even or odd. This behavior can be made understandable by the simple device of decomposing each collision event into two parts; "diametric" scattering through an angle π from a given point on the Fermi surface to the corresponding antipodal point, and subsequent small-angle scattering through an angle of order Θ_2. By virtue of a symmetry property of the spherical harmonics,

$$Y_{LM}(\pi - \theta, \varphi + \pi) = (-1)^L Y_{LM}(\theta, \varphi), \tag{10}$$

it is clear that diametric scattering has no effect when L is even; only the small-angle component is effective in relaxing the distribution. It is thus not surprising to find that for even L, the τ_L decrease initially according to (7). When L is odd, however, diametric scattering is highly effective, and the first few odd τ_L are smaller than τ_2 by a factor of order Θ_2^2. As L increases, the even and odd τ_L approach each other until in the catastrophic limit $\tau_L \sim 2\tau_1$ for all sufficiently large L.

The attenuation may now be computed by substituting (5) or (9) into (3). Either ql_1 or ql_2 may be used as the independent variable and index of nonlocality. Although ql_1 does not appear in the expressions (3) for the attenuation, it is convenient to use it when resistivity and attenuation are to be compared. The result for shear waves[11] is

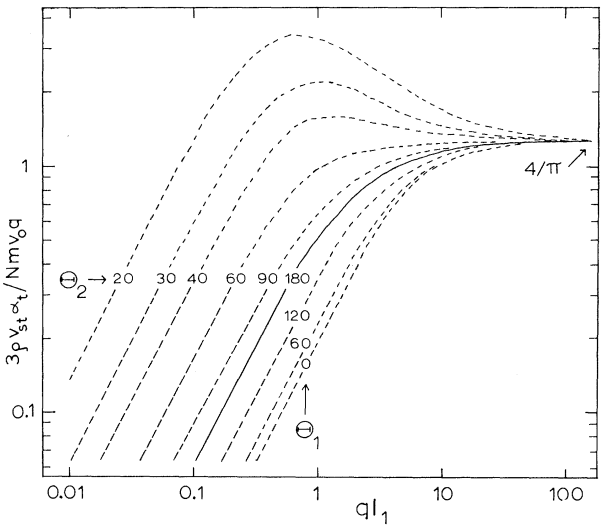

FIG. 1. Calculated shear-wave attenuation as a function of ql_1 showing Pippard function for isotropic scattering (solid curve), some results for forward scattering (dashed curves below solid curve), and some results for backward scattering (dashed curves above solid curve). The parameters Θ_1 and Θ_2 are defined in the text.

shown in Fig. 1. In this diagram, the solid curve is the Pippard function for isotropic scattering. The primary reason for deviations from the Pippard function is the spread in values of the ratio τ_2/τ_1, which is less than unity for forward scattering (reaching the value $\frac{1}{3}$ in the diffusive limit $\Theta_1 \to 0$) and greater than unity for backward scattering (increasing without limit as Θ_2^{-2}). The effects of spread in τ_2/τ_1 may be eliminated by plotting the attenuation as a function of ql_2, as shown in Fig. 2. This representation is more natural for internal comparison of attenuation data, since all models then coincide in the local limit as well as in the nonlocal limit.[12] In Fig. 2, the small deviations from the Pippard function due to forward scattering are insignificant (the scale being logarithmic) and are not shown. Backward scattering gives rise to a pronounced anomaly, however; this takes the form of a peak at intermediate values of ql_2 and a delayed approach to the nonlocal limit.[13] The effect is diluted when backward and forward scattering are combined [by mixing together terms like (5) and (9) in various proportions], but does not disappear until the relative proportion of backward scattering is quite small (this last calculation is not illustrated). The phenomenon is, of course, directly attributable to the disparity in magnitude between the first few even and odd τ_L.

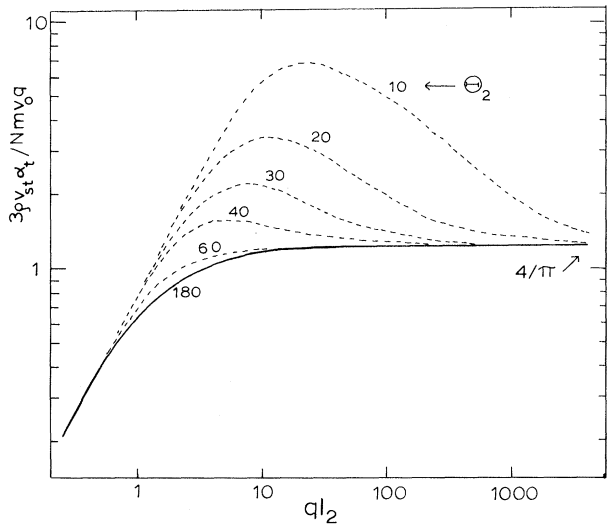

FIG. 2. Calculated shear-wave attenuation as a function of ql_2 showing Pippard function (solid curve) and anomalous peak due to backward scattering. The parameter Θ_2 is defined in the text. Similar results obtain for longitudinal waves.

Among the real metals, the most likely candidates for the observation of the predicted effect would seem to be the alkalis. These metals have nearly spherical Fermi surfaces, and in the repeated zone scheme the neighboring surfaces are close enough to ensure that any scattering mechanism with a strong forward lobe is likely to have a strong backward lobe as well, as a result of umklapp processes. This situation applies for low-temperature electron-phonon scattering, except at the very lowest temperatures (about 1°K in potassium) when umklapp processes are frozen out. The parameter Θ_2 defined in (8) thus corresponds roughly to the angular size of the "hot spots" on the Fermi surface, where umklapp processes are most probable. However, our rather crude model can easily be improved upon; for potassium the machinery for calculating τ_2/τ_L using realistic phonon spectra and electron wave functions has in fact already been set up,[4,5] but the computation of the attenuation as a function of temperature and frequency using (3) is a numerical exercise which might be worth the effort. The result should depend on the particular pseudopotentials used; comparison between such detailed theory and experiment might thus prove to be particularly revealing.

In the case of the alkalis, there are experimental difficulties in determining the total attenuation absolutely. A possible approach is to use the fact that a strong magnetic field can remove some or all of the electronic attenuation[14] (for longitudinal waves an oblique field is required[15]). Some suggestive results have recently been obtained in cesium[16] for which the shear-wave attenuation was found to scale less than linearly with frequency. This could be interpreted as a delayed approach to the nonlocal limit, similar to that shown by some of the model calculations in Fig. 2. In superconductors, of course, the attenuation can be measured more easily, but in all known superconductors other real-metal effects are likely to complicate the picture. Nevertheless, peculiarities in the scattering mechanism can no longer be ruled out as the possible source of ultrasonic anomalies.

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[7]The derivation of (3) is to be presented elsewhere. It should also be derivable using the formalism of Ref. 2, but we have not verified this.

[8]Specifically, (3a) and (3b) in this case are equivalent to Eqs. (23) and (14), respectively, in Ref. 6.

[9]Note that (3a) and (3b) in this case are much more obviously alike in mathematical structure than are Pippard's closed forms.

[10]See, for example, H. Jeffreys and B. S. Jeffreys, *Mathematical Physics* (Cambridge Univ. Press, New York, 1956), 3rd ed., p. 638.

[11]Shear and longitudinal waves show almost identical behavior in this study; the former have been used for illustration.

[12]As pointed out in Ref. 2, Pippard's "ql" thus corresponds most closely to ql_2 in the more general scattering model.

[13]The τ_L were evaluated in Ref. 2 for a screened Coulomb potential and for normal electron-phonon scattering in the Bloch approximation. The calculated attenuation does not deviate significantly from the Pippard expression. It is to be noted that in neither case does $W(\theta)$ have a pronounced backward lobe, but that umklapp scattering might effectively produce one.

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