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Influence of Ion-Resonance Broadening on the Anomalous Heating and Momentum Transfer in a Current-Carrying Plasma

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The anomalous heating and momentum transfer in a current-carrying plasma are calculated including the effects of ion resonance broadening on the nonlinear development of the ion acoustic instability. In the constant-current regime, the anomalous resistivity is found to be of order $(T_i/T_e)^{3/2}\omega_{pe}^{-1}$, which compares favorably with the measured resistivity in several experiments.

The ion acoustic instability has been extensively investigated in the literature[1-11] and a variety of nonlinear mechanisms have been proposed for its stabilization.[5-11] Nonlinear Landau damping by the ions,[5,6] although the most *popular* mechanism invoked for saturation of the wave amplitudes, lacks experimental credibility as the dominant nonlinear process. For example, experimental values of anomalous resistivity associated with the ion acoustic instability[1-3] are typically 1 or 2 orders of magnitude *lower* than the Sagdeev resistivity,[6,12] $\eta_S = 4\pi(T_e/T_i)(m_eV_d^2/T_e)^{1/2}\omega_{pe}^{-1}$, which is calculated under the assumption that nonlinear ion Landau damping is the dominant saturation mechanism. In this Letter, the anomalous heating and momentum transfer in a current-carrying plasma are calculated including the effects of *ion resonance broadening*[7,8] on the nonlinear development of the ion acoustic instability. We consider a uniform, collisionless, unmagnetized plasma consisting of hot electrons drifting with velocity $\vec{V}_d = V_d\hat{e}_z$ relative to a cold-ion background ($T_i \ll T_e$). The electron current is sustained by a *weak* dc electric field $\vec{E}_0 = E_0\hat{e}_z$, which results in the generation of ion acoustic turbulence in the system. After an initial stage of linear growth, the field level saturates as a result of ion resonance broadening, and the system enters a constant-current regime which lasts for several hundred ω_{pe}^{-1}. The anomalous resistivity in the constant-current regime is found to be of order $(T_i/T_e)^{3/2}\omega_{pe}^{-1}$ [see Eq. (7)], which is considerably smaller than the Sagdeev resistivity η_S and compares favorably with the measured resistivity in several lab experiments.[1-3] A detailed study[7,13] of the wave kinetic equation shows that the nonlinear damping produced by ion resonance broadening dominates nonlinear ion Landau damping for temperature ratios $T_i/T_e \gtrsim 0.03$.

The turbulence model adopted in the present analysis consists of the quasilinear kinetic equations generalized to include the effects of ion resonance broadening.[7,14] For simplicity, nonlinear Landau damping and other nonlinear processes are consistently neglected. The spatially averaged jth component distribution function $f_j(\vec{v}, t)$ evolves according to

$$\frac{\partial f_j}{\partial t} + \frac{e_j\vec{E}_0}{m_j}\cdot\frac{\partial f_j}{\partial \vec{v}} = \frac{\partial}{\partial \vec{v}}\cdot\left[\overleftrightarrow{D}_j(\vec{v}, t)\cdot\frac{\partial f_j}{\partial \vec{v}}\right],$$

where

$$\overleftrightarrow{D}_j \simeq (8\pi e_j^2/m_j^2)\int d^3k\,\vec{k}\vec{k}k^{-2}(\omega - \vec{k}\cdot\vec{v})^{-2}\gamma\mathcal{E}_{\vec{k}}$$

in the nonresonant ($\omega \neq \vec{k}\cdot\vec{v}$) region of velocity space,

$$\overleftrightarrow{D}_j \simeq (8\pi e_j^2/m_j^2)\int d^3k\,\vec{k}\vec{k}k^{-2}R_j\mathcal{E}_{\vec{k}} \equiv \overleftrightarrow{D}_j^{\,r}$$

in the resonant ($\omega \simeq \vec{k}\cdot\vec{v}$) region of velocity space,

$$R_j = \mathrm{Re}\int_0^\infty dt\,\exp[i(\vec{k}\cdot\vec{v} - \omega)t - \tfrac{1}{3}\vec{k}\cdot\overleftrightarrow{D}_j^{\,r}\cdot\vec{k}t^3], \quad j = i,$$

$$R_j = \pi\delta(\omega - \vec{k}\cdot\vec{v}), \quad j = e.$$

The spectral energy density associated with the electric field fluctuations, $\mathcal{E}_{\vec{k}}(t)=\langle|\delta E_{\vec{k}}(t)|^2\rangle/8\pi$, satifies $\partial\mathcal{E}_{\vec{k}}/\partial t=2\gamma\mathcal{E}_{\vec{k}}$, where

$$\gamma=\sum_j\gamma_j\simeq\sum_j\frac{\omega^3}{2\omega_{pi}^2}\frac{\omega_{pj}^2}{k^2}\int d^3v\,R_j\vec{k}\cdot\frac{\partial f_j}{\partial\vec{v}}$$

is the growth rate, $\omega^2=\omega_{pi}^2[1+(k^2\lambda_D^2)^{-1}]^{-1}$ is the oscillation frequency squared, λ_D is the electron Debye length, $\omega_{pj}=(4\pi n_je_j^2/m_j)^{1/2}$ is the jth component plasma frequency, and the normalization of f_j is $\int d^3v f_j=1$. To a first approximation, we assume that the spatially averaged distribution functions maintain a Maxwellian structure,

$$f_j(\vec{v},t)=(m_j/2\pi T_j)^{3/2}\exp[-m_j(\vec{v}-\vec{V}_j)^2/2T_j],$$

where $\vec{V}_j(t)=V_j(t)\hat{e}_z$ and $T_j(t)$ vary adiabatically with time in response to the fields. Recent computer simulation experiments by Orens[15,16] indicate that this is a good approximation for about $2000\omega_{pe}^{-1}$ provided the applied electric field is sufficiently weak that $e|E_0|/m_e\omega_{pe}(T_e/m_e)^{1/2}<10^{-3}$. Correct to $O(m_e/m_i)$, we approximate $V_i\simeq 0$ and $V_d=V_e-V_i\simeq V_e$. Assuming $T_e\gg T_i$ and $T_i/m_i\ll T_e/m_e$, the phase velocities of the unstable ion acoustic waves satisfy $T_i/m_i\ll\omega^2/k_z^2\lesssim V_d^2$.

To determine the bulk response of the system to the unstable field fluctuations we compute the velocity moments of the kinetic equation for f_j corresponding to the mean velocity, $V_j=\int d^3v\,v_zf_j$, and kinetic energy density relative to the mean, $K_j=\frac{1}{2}n_jm_j\int d^3v\,(\vec{v}-\vec{V}_j)^2f_j$. After some straightforward algebra that makes use of the approximations enumerated in the previous paragraph, we find

$$\dot{V}_d=-\bar{\nu}_cV_d-(e/m_e)E_0, \tag{1}$$

$$\dot{K}_e=n_em_eV_d^2\int d^3k\,\nu_c(\vec{k})(1-\omega/\vec{k}\cdot\vec{V}_d)+\int d^3k\dot{\mathcal{E}}_{\vec{k}}/k^2\lambda_D^2, \tag{2}$$

$$\dot{K}_i=n_em_eV_d^2\int d^3k\,\nu_c(\vec{k})(\omega/\vec{k}\cdot\vec{V}_d)-\int d^3k\,\dot{\mathcal{E}}_{\vec{k}}(1+1/k^2\lambda_D^2), \tag{3}$$

where

$$\nu_c(\vec{k})\equiv\frac{2\omega_{pe}}{k^2\lambda_D^2}\frac{k_z^2}{k^2}\left[1-\frac{\omega}{\vec{k}\cdot\vec{V}_d}\right]\times\left[(\tfrac{1}{2}\pi)^{1/2}\frac{k\lambda_D}{n_eT_e}\mathcal{E}_{\vec{k}}-\frac{\dot{\mathcal{E}}_{\vec{k}}}{\omega_{pe}n_eT_e}\right], \tag{4}$$

and $\bar{\nu}_c=\int d^3k\,\nu_c(\vec{k})$ is the effective collision frequency for momentum transfer due to the instability. In Eqs. (1)–(4), $-e$ is the electron charge, m_e is the electron mass, $n_j=$ const is the ambient density for component j, and $K_j=3n_jT_j/2$ since f_j is approximated by a Maxwellian. Equations (1) through (3) describe the time rate of change of the average plasma properties. To complete the description, Eqs. (1)–(4) must be supplemented by the kinetic equation for the waves, which can be expressed in the form

$$\dot{\mathcal{E}}_{\vec{k}}=2(\gamma_e^L+\gamma_i^L+\Delta\gamma_i^{NL})\mathcal{E}_{\vec{k}}. \tag{5}$$

In Eq. (5),

$$\gamma_j^L=\frac{\pi\omega^3\omega_{pj}^2}{2\omega_{pi}^2k^2}\int d^3v\,\vec{k}\cdot\frac{\partial f_j}{\partial\vec{v}}\delta(\omega-\vec{k}\cdot\vec{v}),\quad j=i,e,$$

are the *linear* growth (damping) rates associated with component j;

$$\Delta\gamma_i^{NL}=(\omega^3/2k^2)\int d^3v\,R_i\vec{k}\cdot\partial f_i/\partial\vec{v}-\gamma_i^L$$

is the *nonlinear* damping produced by ion resonance broadening.

If the system achieves a *quasisteady* state characterized by $\dot{V}_d\simeq 0\simeq\dot{\mathcal{E}}_{\vec{k}}$, it follows from Eqs. (1) and (5) that $V_d=-eE_0/m_e\bar{\nu}_c$, and $\gamma_e^L+\gamma_i^L+\Delta\gamma_i^{NL}=0$. The balance between linear growth and nonlinear damping determines the level of field fluctuations at saturation. Approximating $R_i=\pi/2\Delta_i$ for $|\vec{k}\cdot\vec{v}-\omega|<\Delta_i=(\vec{k}\cdot\overleftrightarrow{D}_i^r\cdot\vec{k}/3)^{1/3}$, and $R_i=0$ otherwise, and making order-of-magnitude estimates of the various terms in $\gamma_e^L+\gamma_i^L+\Delta\gamma_i^{NL}=0$, it can be shown[7] that the approximate field level at saturation is

$$\frac{\mathcal{E}_F}{n_eT_e}\simeq\frac{3}{2\pi}\left[\frac{T_i}{T_e}\right]^{3/2}k_0^3\lambda_D^3(\Delta k)\lambda_D. \tag{6}$$

In Eq. (6), $\mathcal{E}_F=\int d^3k\,\mathcal{E}_{\vec{k}}$ is the total field energy density, T_i/T_e is the ratio of ion to electron temperature at saturation, Δk is the characteristic width of the excited spectrum in k space, and k_0 is the wave number characteristic of the microturbulence. Making use of Eqs. (4), (6), and $\bar{\nu}_c=\int d^3k\,\nu_c(\vec{k})$, the effective anomalous resistivity under quasisteady-state conditions is $\eta_{an}=E_0/(-n_eeV_d)=4\pi\bar{\nu}_c/\omega_{pe}^2$ for $\dot{V}_d\simeq 0\simeq\dot{\mathcal{E}}_{\vec{k}}$. Solving for η_{an} gives

$$\eta_{an}\simeq 6\frac{(2\pi)^{1/2}}{\omega_{pe}}\left[\frac{T_i}{T_e}\right]^{3/2}\left[1-\frac{\omega_0}{\vec{k}_0\cdot\vec{V}_d}\right]\times(\Delta k)\lambda_Dk_0^2\lambda_D^2, \tag{7}$$

where $\omega_0=\omega(k_0)$. As an example, if $T_i/T_e=\frac{1}{10}$, $C_s/V_d=\omega_{pi}\lambda_D/V_d=\frac{1}{2}$, $k_0^2\lambda_D^2=\frac{1}{2}$, and $(\Delta k)\lambda_D=1$, then $\eta_{an}\simeq 0.1\omega_{pe}^{-1}$.

To obtain a detailed description of the time development of the system, Eqs. (1)–(5) have

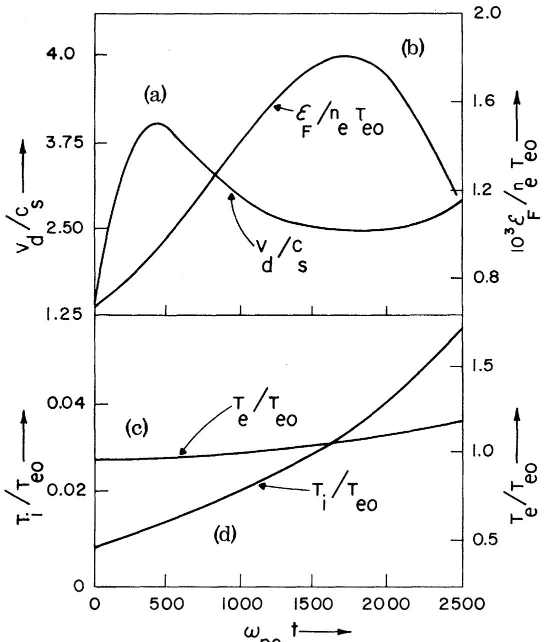

FIG. 1. Time history of (a) $V_d/C_{s0}=V_d(T_{e0}/m_i)^{-1/2}$, (b) $\mathcal{E}_F/n_eT_{e0}=\int d^3k\mathcal{E}_{\vec{k}}/n_eT_{e0}$, (c) T_e/T_{e0}, and (d) T_i/T_{i0}, predicted by Eqs. (1)–(5) for a hydrogen plasma. The initial conditions are $-eE_0[m_e\omega_{pe}(T_{e0}/m_e)^{1/2}]^{-1}=5\times10^{-4}$, $T_{i0}/T_{e0}=\frac{1}{100}$, $\mathcal{E}_F(t=0)/n_eT_{e0}=7\times10^{-4}$, and $V_d(t=0)(T_{e0}/m_e)^{-1/2}=0.0375$, where $T_{i0}\equiv T_i(t=0)$ and $T_{e0}\equiv T_e(t=0)$.

been integrated numerically. For simplicity, a one-dimensional $\vec{k}$ spectrum is considered with $\vec{k}=k\hat{e}_z$. Figure 1 shows the results obtained for a typical set of initial plasma parameters, and $E_0<0$. In Fig. 1, seven modes (k values) have been included in the numerical analysis. Since $E_0<0$, there is an initial increase in V_d in response to the applied field E_0. Once the ion acoustic instability is operative, there is an increase in the level of field fluctuations [Eq. (5)] and a corresponding heating of the electrons and ions [Eqs. (2) and (3)]. This results in a resistance to the flow of electrons [a decrease in V_d according to Eq. (1)], and a simultaneous decrease in the total growth rate [Eq. (5)]. The system approaches a *quasisteady* state characterized by $\dot{V}_d\simeq0\simeq\dot{\mathcal{E}}_k$. For $-eE_0[m_e(T_{e0}/m_e)^{1/2}\omega_{pe}]^{-1}\lesssim10^{-3}$, V_d attains a steady value typically in the range $2.5C_s$ to $5C_s$, and the current remains constant for several hundred ω_{pe}^{-1}.

We note from Fig. 1 that the continued ion and electron heating eventually results in a damping of the waves [since $\gamma_e^L+\gamma_i^L+\Delta\gamma_i^{NL}$ turns negative in Eq. (5)]. Once the fluctuation level is sufficiently low, the electrons again accelerate in response to the applied field E_0 [Eq. (1)], and V_d begins to increase at $\omega_{pe}t\simeq2100$ in Fig. 1. By this time, however, distortions in the distribution due to runaway electrons cannot be neglected,[15] and Eqs. (1)–(4) no longer constitute a valid description. However, since energy is continually supplied to the system by the applied field E_0, it is indeed plausible that V_d will continue to increase to a sufficiently large value that the ion acoustic instability is triggered once again.

The present theory, which utilizes ion resonance broadening as the quasisaturation mechanism, predicts the existence of a quasisteady state in which the current remains constant for several hundred ω_{pe}^{-1} and the corresponding anomalous resistivity is given by Eq. (7). The measured value of anomalous resistivity in recent computer simulation experiments by Orens[15] is in good agreement with the theoretical expression for η_{an}. Furthermore, the qualitative features of the time development of V_d, $\mathcal{E}_F$, T_e, and T_i are similar to those shown in Fig. 1. As an example, for the initial conditions $T_{i0}/T_{e0}=\frac{1}{100}$ and $-eE_0[m_e\omega_{pe}(T_{e0}/m_e)^{1/2}]^{-1}=5\times10^{-4}$, the simulation experiments give $T_i/T_e\simeq\frac{1}{25}$, $V_d/C_s\simeq2.5$, and $(\Delta k)\lambda_D\simeq1$ at quasisaturation. The *measured* value of η_{an} is $2\times10^{-2}\omega_{pe}^{-1}$, whereas Eq. (7) gives $\eta_{an}=3\times10^{-2}\omega_{pe}^{-1}$. The computer simulation experiments also show the development of a nonstationary vortex structure in the ion phase space, which is indicative that the statistical trapping of ions plays an important role during the nonlinear stage of the ion acoustic instability. In Table I, the measured values of anomalous resistivity in several lab experiments[1-3] are compared with the analytical expression for η_{an} given in Eq. (7) and with the Sagdeev resistivity η_S.[6,12] In all cases we have taken $(\Delta k)\lambda_D\simeq1$, $k_0^2\times\lambda_D^2\simeq\frac{1}{2}$, and $\omega_0/k_0V_d\simeq C_s/V_d$; T_e/T_i and V_d/C_s are the measured values at saturation (or quasisaturation) of the ion acoustic instability. Given the uncertainty in the choice of $(\Delta k)\lambda_D$ and the fact that T_e/T_i is not accurately known in some of the experiments, we feel that there is relatively good agreement (within a factor of 5) between Eq. (7) and the experimental values of η_{an}, and that this is indicative that ion resonance broadening is indeed a viable nonlinear mechanism for quasisaturation of the ion acoustic instability. The conditions under which ion resonance broadening dominates other nonlinear mechanism, for the saturation of the ion acoustic instability will be discussed in a subsequent article[13] in which the present analysis is extended

TABLE I. Comparison betwen Eq. (7) and measured values of anomalous resistivity.

Experiment	T_e/T_i	V_d/C_s	m_i/m_e	$\eta_{\rm an}\omega_{pe}$ (Experimental)	$\eta_{\rm an}\omega_{pe}$ [Eq. (7)]	$\eta_{\rm S}\omega_{pe}$ (Sagdeev[a,b])
Wharton *et al.*[c]	5	20	1837	0.6	0.6	3
Mah *et al.*[d]	15[e]	10	40× 1837	0.6	0.1	8
Hirose *et al.*[f]	25[g]	19	40× 1837	0.02	0.05	45
Orens[h] (Computer simulation)	25	2.5	64	0.02	0.03	94

[a]Ref. 7.
[b]Ref. 14.
[c]Ref. 1.
[d]Ref. 2.
[e]Ref. 17.
[f]Ref. 3.
[g]Ref. 18.
[h]Ref. 15.

to include two ion components and a uniform external magnetic field $\vec{B}_0$ parallel to $\vec{E}_0$.

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[16]The simulation experiments referred to here are carried out for $\vec{B}_0 = 0$ and a two-dimensional $\vec{k}$ spectrum. Orens (Ref. 15) finds that $f_j(\vec{v},t)$ maintains a Maxwellian structure adiabatically in time during both the linear growth and nonlinear saturation phase of the instability, before runaway effects become significant. A detailed study (Ref. 13) of the kinetic equations shows that the neglect of non-Maxwellian distortions in $f_j(\vec{v},t)$ for times of order $2000\omega_{pe}^{-1}$ is especially valid in three (or two) dimensions. Depending on the direction of $\vec{v}$ relative to the excited $\vec{k}$ values, the resonant region $(\vec{k}\cdot\vec{v} = \omega)$ of velocity space can extend to infinity, and dramatic changes in $f_j(\vec{v},t)$ (as manifest, for example, by plateau formation over extensive regions of velocity space) do not occur on the time scales considered here.

[17]This value of T_e/T_i is taken from Fig. 2(c) of Ref. 2 at $t \simeq 1.7$ μsec.

[18]The value of T_i was not measured in this experiment. From the observed cutoff frequency and measured value of $V_d(T_e/m_e)^{-1/2}$, it was concluded (Ref. 3) on the basis of quasilinear theory that $T_i/T_e = \frac{1}{2}$ at saturation $(\gamma_e^L + \gamma_i^L \simeq 0)$. However, if ion resonance broadening is included as a stabilization mechanism $(\gamma_e^L + \gamma_i^L + \Delta\gamma_i^{NL} \simeq 0)$, saturation occurs at a *lower* value of T_i/T_e $(\approx \frac{1}{25})$.