Price of Asymptotic Freedom*

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A renormalizable field theory is said to be asymptotically free if the origin of coupling-constant space is an ultraviolet-stable fixed point in the sense of Wilson. Asymptotically free theories are of great interest because they have almost-canonical light-cone singularities, and thus predict phenomena very close to Bjorken scaling. All known examples of asymptotically free theories involve non-Abelian gauge fields. We show that this is not coincidence: No renormalizable field theory without non-Abelian gauge fields can be asymptotically free.

In recent years, the renormalization group of Gell-Mann and Low[1] has played a central part in the investigation of some important asymptotic properties of renormalizable field theories.[2] In particular, the renormalization group is the key to the asymptotic behavior of the coefficient functions in Wilson's operator-product expansion,[3] and the related behavior of electroproduction structure functions in the Bjorken region.[4]

To establish notation, and as an aid to the reader of imperfect memory, let us briefly summarize how this machinery works[5]: In a general renormalizable field theory, let us denote the dimensionless renormalized coupling constants[6] by g^α, and let us define these objects, not as the values of appropriate Green's functions on the mass shell, but as those of the same Green's functions at some point in Euclidean momentum space, characterized by some mass M, large compared to all of the renormalized masses (and dimensionful coupling constants[7]) in the theory. Then the relevant asymptotic properties of the theory are unaffected if we let all the masses (and dimensionful coupling constants) go to zero. Further, in this limit, it is possible to derive differential equations for the changes in the g's as we change the renormalization mass M. These are the differential equations of the renormalization group; they are of the form

$$M\,dg^\alpha/dM = \beta^\alpha(g), \tag{1}$$

where the β's are, in general, functions of all the g's. There exists an algorithm for computing the β's in power series in the g's. If Eqs. (1) admit a family of solutions such that

$$\lim_{M\to\infty} g^\alpha(M) = g_\infty{}^\alpha, \tag{2}$$

we say that $g_\infty{}^\alpha$ is an ultraviolet-stable fixed point. The asymptotic behavior of any theory that lies on a solution of the form (2) is essentially governed by that of the theory at the fixed point. Of special interest is the case where the origin is an ultraviolet-stable fixed point, that is to say, where there exists a family of solutions to the Eqs. (1) such that

$$\lim_{M\to\infty} g^\alpha(M) = 0. \tag{3}$$

In this case, we say the theory is asymptotically free. The coefficient functions in the operator-product expansion display canonical scaling behavior, except for occasional logarithmic factors, which are readily computable in closed form for any given theory. Because of the strong experimental evidence for canonical light-cone behavior, asymptotic freedom is obviously a highly desirable feature for any field theory of the strong interactions. In fact, it has recently been shown, for a wide class of renormalizable field theories, that Bjorken scaling implies asymptotic freedom.[8] [To be honest, we must admit that these theories are interesting for other reasons: To investigate asymptotic freedom, it sufficies to study the behavior of the β functions near the origin, i.e., in lowest-order perturbation theory (one-loop approximation). Thus we have only a problem in coupled nonlinear differential equations, not one in strong-interaction dynamics to boot.]

Until recently, there were *no* known examples of asymptotically free renormalizable field theories. Now there are many.[9] All the known examples involve non-Abelian gauge fields (Yang-Mills fields). The purpose of this note is to show that this is not an accident: *No* renormalizable field theory without non-Abelian gauge fields can be asymptotically free.

Thus, *if* we accept Bjorken scaling as evidence for asymptotic freedom, *then* any acceptable field theory of the strong interactions must involve non-Abelian gauge fields.[10] The strongest interactions (like the weak, though for completely different reasons) must be described by a spontaneously broken gauge field theory. This is a complete reversal of the accepted wisdom of only a few years ago: Then, we believed that in the (Euclidean) asymptotic region, the weak interactions became strong; now, we believe that the strong interactions become weak.

This completes the sermon. We now turn to the proof of the stated result. We begin by recapitulating several partial versions of the result already present in the literature.

(1) If the theory contains Abelian Gauge fields, the wave-function and charge renormalizations of the gauge field only involve the gauge-field coupling, in the one-loop approximation. Thus the renormalization-group equation for the gauge-field coupling is essentially the same as that in quantum electrodynamics, which has been known since the first papers on the renormalization group[1] not to be asymptotically free. Thus the theory can contain no Abelian gauge fields.

(2) If we assume the theory contains *only* spinless mesons, it is easy to show it cannot be asymptotically free.[11] Let us assume the quartic form $\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l$ is positive, where the λ's are the renormalized coupling constants, and the sum on repeated indices is implied. In particular, this implies that λ_{1111} is positive. However, it is easy to compute that

$$M\, d\lambda_{1111}/dM \propto \lambda_{11rs}\lambda_{11rs} \geq 0. \tag{4}$$

Thus the theory cannot be asymptotically free. If we assume the quartic form goes to zero (asymptotic freedom) as M increases, but is not positive,[12] then an application of the methods of Coleman and Weinberg[13] shows directly that the energy of the system cannot be bounded below, and the theory is nonsense.

(3) Thus we are left with theories of spin-$\frac{1}{2}$ fermions and spinless mesons. In this case there exists a partial result due to Zee.[11] Zee shows that the theory cannot be asymptotically free if one assumes that (a) the fermions transform according to a single irreducible representation of some simple Lie group, (b) the mesons transform according to the adjoint representation of the same group, and (c) the group allows only one invariant Yukawa coupling. [Thus, for SU(3), octet mesons coupled to octet fermions obey the first two conditions but violate the third.] Our result can thus be thought of as an extension of Zee's to a much more general case.

We now analyze this case in more detail. In the one-loop approximation, the quartic meson couplings do not contribute to any wave-function renormalizations, nor to the charge renormalization for the Yukawa coupling. Thus, at least to begin with, we may study the renormalization-group equations for the Yukawa couplings alone.[14] For orientation, we begin with a single scalar meson coupled to a single fermion: $\mathcal{L}'=g\bar{\psi}\psi\varphi$. A simple computation shows that Eq. (1) takes the form

$$M\, dg/dM = (g^3/16\pi^2)(2+\tfrac{1}{2}+\tfrac{1}{2}+2). \tag{5}$$

The four terms on the right-hand side of this equation represent the effects of meson wave-function renormalization [Fig. 1(a)], wave-function renormalization of each of the two nucleons [Figs. 1(b) and 1(c)], and charge renormalization [Fig. 1(d)].[15] It follows directly from Eq. (5) that

$$M\, dg^2/dM = 5g^4/8\pi^2 \geq 0. \tag{6}$$

Hence, the theory is not asymptotically free.

We now turn to the theory of an arbitrary number of mesons and fermions, with the most general (not necessarily parity conserving) Yukawa coupling. Let the interaction be $\mathcal{L}'=\bar{\psi}^a(A_{ab}{}^i + iB_{ab}{}^i\gamma_5)\psi^b\varphi^i$, where we have taken the meson fields to be real and the sum over repeated indices is implied. (We use a Hermitian γ_5.) Reality of the Lagrangian implies that the A's and B's are Hermitian matrices. It is convenient to consider these couplings as forming a set of square matrices g^i:

$$g_{ab}{}^i = A_{ab}{}^i + iB_{ab}{}^i. \tag{7}$$

We can now read off from Fig. 1 the generalization of Eq. (5) [note that since γ_5 anticommutes with the propagator of a massless Fermi field,

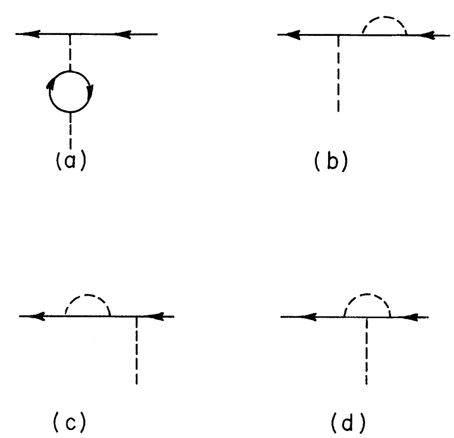

FIG. 1. The four one-loop graphs that contribute to the β functions for Yukawa couplings. Directed lines, fermions; dashed lines, spinless mesons.

pushing a Yukawa coupling through a propagator in Fig. 1 has the effect of turning g into $g^\dagger$]:

$$16\pi^2 M\, dg^i/dM = (\mathrm{Tr} g^i g^{j\dagger})g^j + (\mathrm{Tr} g^{i\dagger} g^j)g^j + \tfrac{1}{2} g^i g^{j\dagger} g^j + \tfrac{1}{2} g^j g^{j\dagger} g^i + 2 g^j g^{i\dagger} g^j. \tag{8}$$

Whence, in analogy to Eq. (6),

$$8\pi^2 M (d/dM)(\mathrm{Tr} g^{i\dagger} g^i) = \mathrm{Tr} g^i g^{j\dagger}\, \mathrm{Tr} g^{i\dagger} g^j + \mathrm{Tr} g^i g^{j\dagger}\, \mathrm{Tr} g^i g^{j\dagger} + \tfrac{1}{2} \mathrm{Tr} g^i g^{i\dagger} g^j g^{j\dagger} + \tfrac{1}{2} \mathrm{Tr} g^{i\dagger} g^i g^{j\dagger} g^j + 2\, \mathrm{Tr} g^i g^{j\dagger} g^i g^{j\dagger}. \tag{9}$$

Note that the differentiated object in this equation is a positive quantity that must go to zero as M goes to infinity, if the theory is to be asymptotically free. Now the central pair of terms are obviously positive, being the trace of the square of a Hermitian matrix. Also, the first term is greater than the second, for

$$\mathrm{Tr} g^i g^{j\dagger}\, \mathrm{Tr} g^i g^{j\dagger} = \mathrm{Re}(\mathrm{Tr} g^i g^{j\dagger}\, \mathrm{Tr} g^i g^{j\dagger}) \leqslant (\mathrm{Tr} g^i g^{j\dagger})(\mathrm{Tr} g^i g^{j\dagger})^*. \tag{10}$$

Finally,

$$2\, \mathrm{Tr} g^i g^{j\dagger}\, \mathrm{Tr} g^i g^{j\dagger} + 2\, \mathrm{Tr} g^i g^{j\dagger} g^i g^{j\dagger} = (g_{ab}{}^i g_{cd}{}^i + g_{ad}{}^i g_{cb}{}^i)(g_{ba}{}^{j\dagger} g_{dc}{}^{j\dagger} + g_{da}{}^{j\dagger} g_{bc}{}^{j\dagger}) \geqslant 0. \tag{11}$$

Thus,

$$M(d/dM)(\mathrm{Tr} g^{i\dagger} g^i) \geqslant 0, \tag{12}$$

and the theory cannot be asymptotically free.

This completes the analysis in the one-loop approximation. Are there any graphs with more loops that can change our conclusions? It is easy to see that the only graph that can possibly compete with those we have considered is that shown in Fig. 2. This graph is of order $g\lambda^2$, and could compete with the $O(g^3)$ graphs of Fig. 1, if λ were on the order of g. All other graphs, though, are either proportional to g^3, times powers of coupling constants, or proportional to $g\lambda^2$, times powers of coupling constants, and therefore can be safely neglected in the region of small coupling constants.

It is easy to compute the effects of this graph. It adds an extra term to the right-hand side of Eq. (8):

$$16\pi^2 M\, dg^i/dM = \cdots + M^{ij} g^j, \tag{13}$$

where

$$M^{ij} = \lambda^{irst}\lambda^{jrst}/48\pi^2. \tag{14}$$

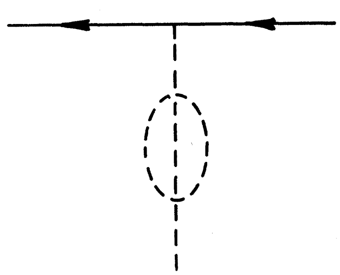

FIG. 2. The only two-loop graph that can possibly compete with the graphs of Fig. 1.

Note that M is a positive-definite matrix. This addition produces an addition to Eq. (9):

$$8\pi^2 M(d/dM)(\mathrm{Tr} g^{i\dagger} g^i) = \cdots + M^{ij}\, \mathrm{Tr} g^{i\dagger} g^j. \tag{15}$$

However, this is positive, as is seen by going to the frame in which M is diagonal; thus, it does not change the inequality (12). This completes the proof.

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[5]For a more detailed description, see S. Coleman, in Proceedings of the International School of Subnuclear Physics "Ettore Majorana," Erice, Italy, 8–26 July 1971 (to be published).

[6]That is to say, quartic couplings of spinless mesons, Yukawa couplings, and gauge-field couplings.

[7]That is to say, cubic and linear couplings of spinless mesons.

[8]C. Callan and D. Gross, to be published.

[9]D. Gross and F. Wilczek, Phys. Rev. Lett. 26, 1343 (1973); H. Politzer, Phys. Rev. Lett. 26, 1346 (1973). The asymptotic freedom of pure Yang-Mills theory was

known to G. 't Hooft in 1972, but not published by him. K. Symanzik and M. Veltman, private communication.

[10]We emphasize the *if*. We are well aware that there are many alternative explanations of Bjorken scaling in the literature, which the reader may well find more plausible than this one.

[11]A. Zee, to be published.

[12]This is the case considered by K. Symanzik, "A Field Theory with Computable Large-Momentum Behavior" (to be published).

[13]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[14]The reverse is not true. The Yukawa couplings enter the equations for the quartic couplings; indeed, the box diagram makes a negative contribution to Eq. (4) and spoils the inequality.

[15]We remind the reader that β is proportional to the coefficient of the logarithmic divergence in the sum of these graphs, after mass-renormalization subtractions have been made. Also, for the reader who wishes to reproduce the computation, the first three graphs have one half their naive value, essentially because it is the square root of the wave-function renormalization constant that enters the renormalization-group equations.

Positivity and the Axial-Vector Current in Quantum Electrodynamics*

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In finite theories of quantum electrodynamics, positivity implies $d>3$ for the dimension d of an axial-vector current $J_{5\mu}$ with nonzero anomaly. This result is not contradicted in the Johnson-Baker-Willey and Adler models: Arguments for the neglect of internal fermion creation and annihilation fail for $J_{5\mu}$ amplitudes because illegal skeleton expansions are involved.

Attemps[1,2] to include an axial-vector current $J_{5\mu}$ in finite theories[3-5] of quantum electrodynamics (QED) have produced unexpected difficulties. In particular, the annihilation condition[6]

$$J_\mu(0)|0\rangle = 0 \quad (\text{fermion mass } m=0) \tag{1}$$

for the electromagnetic current J_μ seems to imply the result[1,2]

$$\langle 0|J_\alpha J_\beta J_{5\gamma}|0\rangle = 0 \quad (m=0). \tag{2}$$

However, in any theory summed over gauge-invariant subsets of diagrams (with $m\neq 0$), the corresponding anomalous constant S is not renormalized[7]:

$$\partial^\gamma J_{5\gamma} = J_5 + (\alpha S/4\pi)[F\cdot\tilde{F}],$$

$$S = -\tfrac{1}{12}\pi^2\epsilon^{\mu\rho\alpha\beta}\iint d^4x\, d^4y\, x_\mu y_\rho T \times\langle 0|J_\alpha(x)J_\beta(0)J_5(y)|0\rangle = 1 \quad (m\neq 0) \tag{3}$$

(the symbol J_5 represents a soft pseudoscalar operator, and $[F\cdot\tilde{F}]$ denotes the renormalized gauge-invariant normal-product operator constructed from the electromagnetic field-strength tensor $F_{\alpha\beta}$ and its dual $\tilde{F}_{\alpha\beta}$). According to Wilson's analysis[8] of the anomaly, Eq. (2) implies $S=0$,[9] a result which is not compatible with (3).

We have already given an extensive discussion of this problem and related difficulties elsewhere.[10] This abbreviated version, unencumbered with side issues, serves to emphasize the main conclusions:

(a) Positivity and Eq. (3) imply that d, the dimension of $J_{5\mu}$, is *greater* than 3; in that case, Eqs. (1) and (3) are compatible, and Eq. (2) is incorrect.

(b) In the Johnson-Baker-Willey[4] (JBW) and Adler[5] models, the argument that internal fermion creation and annihilation may be asymptotically neglected cannot be applied to $J_{5\mu}$ amplitudes because it involves the use of an illegal skeleton expansion.

Instead of setting m equal to zero, we consider products of smeared gauge-invariant operators