Anomalous Diffusion in a Magnetized Plasma

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It is shown that the anomalous diffusion of a plasma across a strong external magnetic field may result from the hydrodynamic behavior of the plasma. The hydrodynamic contribution to the velocity-velocity correlation function is found to be proportional to $1/B$ for a magnetized plasma. It is suggested that classical calculations evaluate only what is essentially the microscopic part of the velocity-velocity correlations, and so do not give the $1/B$ contribution.

There has recently been considerable theoretical interest in the diffusion of a plasma across a strong external magnetic field. Early calculations[1] had predicted a $1/B^2$ dependence for the diffusion coefficient; however, experiments have found a $1/B$ dependence for sufficiently strong fields. By use of the guiding-center model for the plasma, the $1/B$ dependence has been derived by Taylor and McNamara[2] and Montgomery and Tappert[3] for a nonturbulent, two-dimensional plasma. These calculations have been generalized to three dimensions by Montgomery, Lui, and Vahala.[4] In addition, computer experiments[5,6] have suggested a hydrodynamic origin for the $1/B$ dependence. Specifically, these computer experiments seem to indicate that so-called "convective cells" are responsible for the anomaly. The purpose of the present paper is to explore from a different point of view the possibility of a hydrodynamic origin for anomalous diffusion.

At present there is no rigorous derivation of a correlation-function expression for the diffusion of a plasma in an external magnetic field. Frequently it is speculated that an equation similar to that found for (a) diffusion in a system with no external magnetic field, or (b) the diffusion of Brownian-motion particles, may be valid.[7] For ambipolar diffusion we will assume that

$$D_{xx}=\lim_{z\to 0^-}\lim_{k\to 0}\frac{\int_{-\infty}^{\infty}e^{i\vec{k}\cdot\vec{x}}\int_0^{\infty}e^{zt}\langle v_x(0,0)v_x(\vec{x},t)\rangle_0\,dt\,d^3x\,\bar{n}^2}{\int_{-\infty}^{\infty}e^{i\vec{k}\cdot\vec{x}}\langle n(0,0)n(x,t)\rangle_0\,d^3x}, \tag{1}$$

where v_x is the x component of velocity, $\bar{n}$ the average number density of the electrons (or ions), and $\langle\ \rangle_0$ represents the equilibrium average. Equations very similar to this are often used to evaluate the diffusion coefficient in a magnetized plasma.[2-4] The order of the limits in Eq. (1) is important: first $k\to 0$ and then $z\to 0$.[7] Throughout this derivation we will assume that the plasma volume is infinite.

Correlation functions such as the one occuring in Eq. (1) may be thought of as being composed of two parts,[8] a microscopic and a hydrodynamic part. The microscopic part describes the relaxation of the system to hydrodynamics, while the hydrodynamic part describes the hydrodynamic behavior itself. Kinetic-theory calculations frequently only determine the microscopic part. This is not a serious problem in most cases for the hydrodynamic part usually gives no net contribution when integrated over time, and when the appropriate limits are taken.[8] In this paper we will calculate the hydrodynamic part of the velocity-velocity correlation function and show that it does give a net $1/B$ contribution.

The procedure that will be followed was originally proposed by Landau and Placzek.[9] It assumes that $v(\vec{x},t)$ in Eq. (1) can be described by the hydrodynamic equations (in our case magnetohydrodynamics). In addition, we will set the wave vector $\vec{k}$ parallel to the external field $\vec{B}$ (it can be shown that $\vec{k}$ perpendicular to $\vec{B}$ gives identical results). The magnetohydrodynamic equations and their eigenvectors and eigenvalues for $\vec{k}$ parallel to $\vec{B}$ have already been determined.[10] Using them we can write

$$v(t)=\sum_j\varphi_j\exp(-P_jt), \tag{2}$$

where φ_j is the jth eigenvector and P_j is its associated eigenvalue. The φ_j's are each known up to an arbitrary constant. This constant is determined by the requirement that Eq. (2) be valid for $t=0$ and also that the other important hydrodynamic variables (δv_y, δE_x, δE_y, δB_x, and δB_y) satisfy the initial conditions $(\delta v_y, \ldots)=0$ when $t=0$. The eigenvectors from Ref. 10 can then be written as follows:

$$\begin{bmatrix} B_z^2\epsilon_0\beta/2\rho & B_z^2\epsilon_0\beta/2\rho & \frac{1}{4}\beta & \frac{1}{4}\beta & \frac{1}{4}\beta & \frac{1}{4}\beta \\ iB_z^2\epsilon_0\beta/2\rho & -iB_z^2\epsilon_0\beta/2\rho & 0 & 0 & 0 & 0 \\ \frac{1}{2}iB_z\beta & -\frac{1}{2}iB_z\beta & 0 & 0 & 0 & 0 \\ -\frac{1}{2}B_z\beta & -\frac{1}{2}B_z\beta & \frac{1}{4}B_z\beta & \frac{1}{4}B_z\beta & \frac{1}{4}B_z\beta & \frac{1}{4}B_z\beta \\ 0 & 0 & \dfrac{\delta v_0\rho}{4cB_z\epsilon_0} & \dfrac{\delta v_0\rho}{4cB_z\epsilon_0} & \dfrac{-\delta v_0\rho}{4cB_z\epsilon_0} & \dfrac{-\delta v_0\rho}{4cB_z\epsilon_0} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3}$$

where $\beta=\delta v_0(1+B_z^2\epsilon_0/\rho)^{-1}$. The first two columns of Eq. (3) are the transverse electrical conductivity eigenvectors, while the remaining four are the Alfvén eigenvectors. Terms in Eq. (3) of order k or greater have not been included. They would not contribute because of the order of the limits in Eq. (1).

Substituting the eigenvectors from Eq. (3) into Eq. (2) and using the values for P_j from Ref. 10, we get

$$v_x(t)=\tfrac{1}{2}v_x(0)\frac{B_z^2\epsilon_0}{\rho}\left(1+\frac{B_z^2\epsilon_0}{\rho}\right)^{-1}\exp\left[-(\sigma_{xx}+i\sigma_{xy})\left(\frac{B_z^2}{\rho}+\frac{1}{\epsilon_0}\right)t\right] + \tfrac{1}{2}v_x(0)\frac{B_z^2\epsilon_0}{\rho}\left(1+\frac{B_z^2\epsilon_0}{\rho}\right)^{-1}\exp\left[-(\sigma_{xx}-i\sigma_{xy})\left(\frac{B_z^2}{\rho}+\frac{1}{\epsilon_0}\right)t\right]. \tag{4}$$

Using Eq. (4) together with the hydrodynamic part of Eq. (1) we find that

$$D_{xx}{}^h=\tfrac{1}{2}\bar{n}\langle v_x, v_x; 0, t=0\rangle\frac{B_z^2\epsilon_0^2}{\rho}\left(1+\frac{B_z^2\epsilon_0}{\rho}\right)^{-2}\left(\frac{1}{\sigma_{xx}+i\sigma_{xy}}+\frac{1}{\sigma_{xx}-i\sigma_{xy}}\right), \tag{5}$$

where $\langle v, v; 0, t=0\rangle \equiv \int_{-\infty}^{\infty}\langle v(0,0)v(\vec{x}, t=0)\rangle\, d^3x$. Equation (5) is a general expression for the hydrodynamic contribution to the transverse diffusion coefficient. Before we show that Eq. (5) can give a $1/B$ dependence for the diffusion coefficient, let us simplify it by assuming that the contribution from σ_{xy} is negligible.[3] Equation (5) can then be written

$$D_{xx}{}^h=\bar{n}\langle v, v; 0, t=0\rangle(B_z^2\epsilon_0^2/\rho)(1+B_z^2\epsilon_0/\rho)^{-2}\sigma_{xx}^{-1}. \tag{6}$$

It has recently been suggested, on the basis of nonequilibrium thermodynamic arguments,[11] that the electrical conductivity and the particle-diffusion tensors are not independent transport coefficients. This is in agreement with previous results found by Montgomery and Tappert[3] and Vahala[12] for a guiding-center plasma. We can make use of these results[12] to write

$$D_{xx}=(kT/e^2n_e)\sigma_{xx}, \tag{7}$$

where k is the Boltzmann constant, T the temperature, e the electronic charge, and n_e the electron number density. Equation (6) implies that

$$D_{xx}{}^h=\bar{n}\langle v, v; 0, t=0\rangle^{1/2}\frac{B_z\epsilon_0}{\rho^{1/2}}\left(\frac{kT}{e^2n_e}\right)^{1/2}\left[\left(\frac{B_z^2\epsilon_0}{\rho}+1\right)\left(1+\frac{D_{xx}{}^m}{D_{xx}{}^h}\right)^{1/2}\right]^{-1}, \tag{8}$$

where $D_{xx}{}^m$ represents the contribution to the diffusion coefficient arising from the microscopic part of Eq. (1). If the static correlation is given by

$$\langle v, v; 0, t=0\rangle = kT/\rho, \tag{9}$$

then we get as a final result

$$D_{xx}{}^h=\bar{n}^{1/2}kT\frac{B_z\epsilon_0}{\rho}\frac{1}{(e^2n_e)^{1/2}}\left[\left(\frac{B_z^2\epsilon_0}{\rho}+1\right)\left(1+\frac{D_{xx}{}^m}{D_{xx}{}^h}\right)^{1/2}\right]^{-1}, \tag{10}$$

$$D_{xx}=\bar{n}^{1/2}\lambda_D{}^2\left(\frac{e^2}{\epsilon_0 M}\right)^{1/2}\left(\frac{m_e}{M}\right)^{1/2}\left(\frac{\omega_{ce}}{\omega_{pe}}\right)\left\{\left[\frac{m_e}{M}\left(\frac{\omega_{ce}}{\omega_{pe}}\right)^2+1\right]\left(1+\frac{D_{xx}{}^m}{D_{xx}{}^h}\right)^{1/2}\right\}^{-1}+D_{xx}{}^m, \tag{11}$$

where λ_D is the Debye length, m_e the electron mass, M the sum of the electron and ion masses, ω_{pe} the electron plasma frequency, and ω_{ce} the electron cyclotron frequency. For large fields, Eq. (11) has the anomalous $1/B$ behavior. Figure 1 shows the diffusion coefficient plotted against the magnetic field ω_{pe}/ω_{ce} with the same parameters used by Okuda and Dawson[5] (Fig. 1) but assuming $D_{xx}{}^m$ is given by the classical result[1]

$$D_{xx}{}^m = \frac{1}{3\sqrt{2}\pi^{3/2}} \frac{\omega_{pe}{}^2}{\omega_{ce}{}^2 n\lambda_D} \ln(n\lambda_D{}^3\pi)\omega_{pe}\left(\frac{m_i}{M}\right)^{1/2}. \tag{12}$$

When the functional dependence of the diffusion coefficient on the magnetic field found in Eq. (11) is compared with the computer simulations there appears to be general agreement. This is clear when Fig. 1 is compared with Figs. 2 and 3 of Ref. 5. (Note: Different plasma parameters are involved in all of these figures.) In view of the rather small plasma volume that can be treated in the computer simulations, a quantitative comparison between the present results and those of Okuda and Dawson may not be of value. This is particularly so since most equilibrium derivations of anomalous diffusion indicate that the anomalous behavior becomes more important with increasing plasma volume (in fact, diverges in the limit of infinite systems[2,5,6]). However, if a quantitative comparison is made, it is found that Eq. (11) agrees with the computer results only if a multiplicative factor of approximately $\frac{1}{100}$ is included in the hydrodynamic part of the diffusion coefficient.

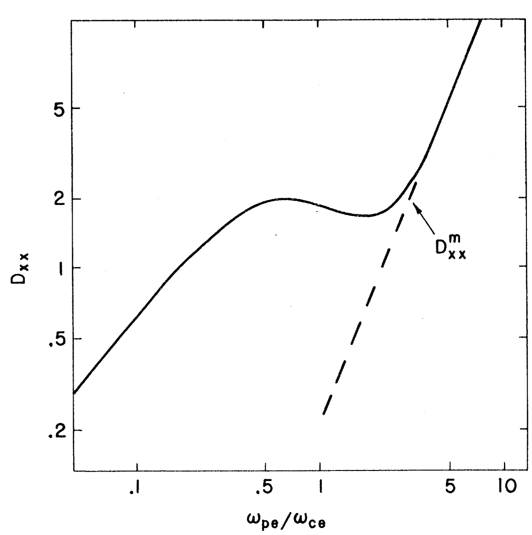

FIG. 1. Diffusion coefficient plotted against magnetic field. For small fields the collision-theory contribution dominates. $\lambda_D = 3$, $m_i/m_e = 1.25$, $n\lambda_D^3 = 3.5$.

In conclusion, the present calculation gives (a) a functional form for the anomalous diffusion that appears to be similar to the computer results, (b) a temperature dependence for the anomalous diffusion in agreement with the Bohm conjecture,[13] and (c) a diffusion coefficient defined for an infinite plasma.

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