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Interaction Barrier in Charged-Particle Nuclear Reactions*

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Simple expressions are obtained for the total reaction cross section in terms of the interaction barrier for the s wave. These expressions allow the interaction barrier to be determined experimentally. Analysis of experimental data for heavy ions on ^{238}U shows that the effective radius parameter decreases as projectile charges increase.

In charged-particle nuclear reactions, it is of interest to measure the height of the barrier between the interacting nuclei. Such a measurement provides information on the fusion process,[1] which is an important intermediate step in the production of superheavy nuclei by heavy-ion reactions. It may also facilitate the study of distortion effects[2-6] and of the dependence of the barrier height on the charges and shapes of the interacting nuclei.[7]

It is known that the probability of penetration is one-half at the top of an inverted harmonic-oscillator potential. It is therefore convenient to define the interaction barrier for the lth partial wave as the energy E_l at which the absorption probability $P(E_l, l)$ is one-half. While such a definition is model independent, it assumes a simple physical meaning in the ingoing-wave strong-absorption model[8] with parabolic barriers.

With such a definition, the barriers can be readily obtained by analyzing the elastic scattering or reaction cross-section data with an optical model or by parametrizing the phase shifts. For a given incident energy E, one finds the value of l_b for which the absorption probability is given by $1-|\eta_{l_b}|^2=\frac{1}{2}$. It can then be said that the interaction barrier for the l_bth partial wave is the incident energy E. If data are available for different energies, the interaction barrier for various values of l can be obtained.

Of particular interest is the interaction barrier for the s wave which is traditionally called the "Coulomb barrier." We wish to present in this article another way to measure this barrier by employing a simple analytic expression for the total reaction cross section obtained in the ingoing-wave strong-absorption model.

We shall consider first two spherical nuclei and the case of no dynamical distortion. Following Thomas,[9] Huizenga and Igo,[10] and Rasmussen and Sugawara-Tanabe,[11] we approximate the various barriers for different partial waves by inverted harmonic-oscillator potentials of height E_l and frequency ω_l. For an energy E, the probability $P(l,E)$ for the absorption of the lth partial wave is then given by the Hill-Wheeler formula[12]

$$P(l,E)=\{1+\exp[2\pi(E_l-E)/\hbar\omega_l]\}^{-1}. \qquad (1)$$

In consequence, the total reaction cross section is

$$\sigma_r(E)=\frac{\pi}{k^2}\sum_l \frac{2l+1}{1+\exp[2\pi(E_l-E)/\hbar\omega]}. \qquad (2)$$

Instead of parametrizing the nuclear interaction in the form of a diffused potential well, as is done in Refs. 9–11, we wish to write E_l and $\hbar\omega_l$ as a function of l directly so that the interaction barrier E_0 enters explicitly. This can be done using a diffuse potential as a guide. The effective potential for the reaction is

$$V(r)=-V_0/\{1+\exp[(r-\mathcal{R}_1-\mathcal{R}_2)/a]\}$$
$$+Z_1Z_2e^2/r+\hbar^2 l(l+1)/2\mu r^2, \qquad (3)$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the potential radii and μ is the reduced mass. The interaction barrier for the lth partial wave is just

$$E_l=V(R_l), \qquad (4)$$

where the radial separation R_l is obtained from the condition

$$[dV(r)/dr]_{R_l}=0. \qquad (5)$$

The frequency ω_l is related to $V(r)$ by

$$\hbar\omega_l = \hbar[d^2V(r)/dr^2|_{R_l}/\mu]^{1/2}. \tag{6}$$

With Eqs. (4)–(6), we calculate as an example E_l, $\hbar\omega_l$, and R_l for the case of $^{16}O + ^{238}U$ using a diffuse well with the parameters of Viola and Sikkeland[13] who treated ^{238}U as spherical: $V_0 = 70$ MeV, $\mathcal{R}_i = 1.25A_i^{1/3}$, and $a = 0.48$ fm. One finds there that R_l and $\hbar\omega_l$ are rather insensitive to l. This result justifies the following parametrization in the region of $l = 0$:

$$E_l \cong E_0 + \hbar^2 l(l+1)/2\mu R_0^2, \tag{7}$$

$$\hbar\omega_l \simeq \hbar\omega_0. \tag{8}$$

Using approximations (7) and (8) and replacing the sum in Eq. (2) by an integral, the reaction cross section can be integrated to yield

$$\sigma_r(E) = (R_0^2\hbar\omega_0/2E)\ln\{1 + \exp[2\pi(E - E_0)/\hbar\omega_0]\}. \tag{9}$$

One observes that for relatively large values of E, the present result reduces to the well-known formula

$$\sigma_r(E) = \pi R_0^2(1 - E_0/E). \tag{10}$$

For relatively small values of E such that $E \ll E_0$, we have

$$\sigma_r(E) = (R_0^2\hbar\omega_0/2E)\exp[2\pi(E - E_0)/\hbar\omega_0]. \tag{11}$$

We turn now to the interaction between two deformed nuclei of radii $\mathcal{R}_i$ with deformation parameters $\beta_2^{(i)}$ and making orientation angles Θ_i with respect to the collision axis. We fix our attention on the case of no dynamical distortion and rotation. The nuclear potential now becomes

$$V_N(r,\theta) = -V_0/(1 + \exp(\{r - \sum_{i=1}^{2}\mathcal{R}_i[1 + (5/4\pi)^{1/2}\beta_2^{(i)}P_2(\cos\theta_i)]\}/a)), \tag{12}$$

and the Coulomb potential becomes[14]

$$V_c(r,\theta) = \frac{Z_1Z_2e^2}{r} + \left(\frac{9}{20\pi}\right)^{1/2}\left(\frac{Z_1Z_2e^2}{r^3}\right)\sum_{i=1}^{2}\mathcal{R}_i^2\beta_2^{(i)}P_2(\cos\theta_i) + \left(\frac{3}{7\pi}\right)\left(\frac{Z_1Z_2e^2}{r^3}\right)\sum_{i=1}^{2}\mathcal{R}_i^2[\beta_2^{(i)}P_2(\cos\theta_i)]^2, \tag{13}$$

where θ_i is the angle measured between the radius vector $\vec{r}$ and the symmetry axis of the ith nucleus. The quadrupole-quadrupole term, which is proportional to $\beta_2^{(1)}\beta_2^{(2)}$, is of shorter range and can therefore be neglected.

We shall take the perturbative approach by considering first the interaction barrier for the case of head-on collisions. By virtue of Eqs. (12) and (13), we write the interaction barrier for head-on collisions in the form

$$E_0(\Theta_1,\Theta_2) = \bar{E}_0 + \sum_{i=1}^{2}f_i(R_0)\beta_2^{(i)}P_2(\cos\Theta_i) + \sum_{i=1}^{2}g_i(R_0)\beta_2^{(i)2}\{[P_2(\cos\Theta_i)]^2 - \tfrac{1}{5}\} + h(R_0)\beta_2^{(1)}\beta_2^{(2)}P_2(\cos\Theta_1)P_2(\cos\Theta_2). \tag{14}$$

Here, an extra term of order β_i^2 is introduced so that $\bar{E}_0$ is the interaction barrier after the orientation angles are averaged. The functions $f_i(R_0)$, $g_i(R_0)$, and $h(R_0)$ can be shown to be

$$f_i(R_0) = (20\pi)^{-1/2}(Z_1Z_2e^2\mathcal{R}_i/R_0^2)(-5 + 3\mathcal{R}_i/R_0), \tag{15}$$

$$g_i(R_0) = -\mathcal{R}_i^2[46Z_1Z_2e^2/7R_0^3 + 5\mu(\hbar\omega_0)^2/\hbar^2]/8\pi, \tag{16}$$

$$h(R_0) = -5\mathcal{R}_1\mathcal{R}_2[2Z_1Z_2e^2/R_0^3 + \mu(\hbar\omega_0)^2/\hbar^2]/4\pi. \tag{17}$$

The angular momentum, l, is now not a good quantum number as the interaction depends on angles. To the extent that such a perturbation on the orbital motion due to nuclear deformation is negligible, one may retain l as an approximate integral of motion and assume the dependence of interaction barri-

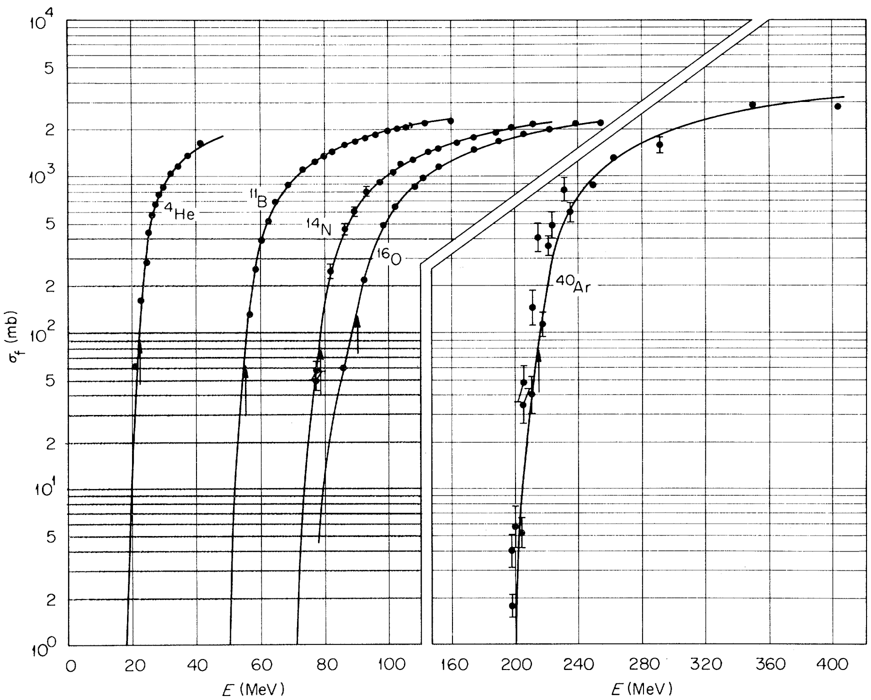

FIG. 1. Total fission cross sections of ^{4}He, ^{11}B, ^{14}N, ^{16}O, and ^{40}Ar on ^{238}U. The solid curves are theoretical fits treating the fission cross sections as total reaction cross sections. The arrows indicate the locations of the interaction barrier $\bar{E}_0$ for the s wave when all orientations are averaged.

er E_l on l for orientation angles Θ_1 and Θ_2 as given similar to Eq. (7):

$$E_l(\Theta_1, \Theta_2) \cong E_0(\Theta_1, \Theta_2) + \hbar^2 l(l+1)/2\mu R_0^2. \tag{18}$$

Upon an integration over l, one obtains then the total reaction cross section for orientation angles Θ_1 and Θ_2:

$$\sigma_r(E, \Theta_1, \Theta_2) = (R_0^2\hbar\omega_0/2E)\ln(1+\exp\{2\pi[E - E_0(\Theta_1, \Theta_2)]/\hbar\omega_0\}). \tag{19}$$

When averaged over the orientations, the total reaction cross section is, up to the second order in β_2,

$$\langle\sigma_r(E)\rangle = \frac{R_0^2}{2}\frac{\hbar\omega_0}{E}\left[\ln(1+e) - \frac{2\pi e}{5\hbar\omega_0(1+e)}\sum_{i=1}^{2}\beta_2^{(i)2}g_i(R_0) + \frac{\pi^2}{5(\hbar\omega_0)^2}\frac{e}{(1+e)^2}\sum_{i=1}^{2}\beta_2^{(i)2}f_i(R)^2\right], \tag{20}$$

where

$$e = \exp\{2\pi[E - \bar{E}_0 + \sum_{i=1}^{2} g_i(R_0)\beta_2^{(i)2}/5]/\hbar\omega_0\}.$$

To show how Eqs. (19) and (20) can be applied, we consider the experimental total fission cross sections[13,15] of various heavy ions on ^{238}U taken as the total reaction cross sections. We limit ourselves to the projectiles ^{4}He, ^{11}B, ^{14}N, ^{16}O, and ^{40}Ar which are treated as spherical. The average over the orientation angles of ^{238}U is performed by integrating Eq. (19) numerically. Search is made of the parameters E_0, $\hbar\omega_0$, and $R_0 \equiv r_{0b}(A_1^{1/3} + A_2^{1/3})$. The other parameters are taken to be $\beta_2(^{238}\mathrm{U}) = 0.277$[16] and $\mathcal{R}_i = 1.2A_i^{1/3}$. As is seen, the fits are very good for the cases considered (Fig. 1). The sets of parameters for the fits are plotted in Fig. 2 as a function of the projectile charge. To facilitate comparison with the usual definition of the Coulomb barrier, we introduce an effective radius parameter r_e by

$$\bar{E}_0 = Z_1Z_2e^2/r_e(A_1^{1/3} + A_2^{1/3}). \tag{21}$$

One finds that r_e decreases with projectile charge, in agreement with recent observations.[1,7] For the parameters $\hbar\omega$ and r_{0b}, Fig. 2 shows interesting trends as well as deviations from the trend. A systematic study of the barrier parameters from all available data undertaken by Alexander and Vaz[17] recently will be of great value in shedding more light on the interaction between two nuclei.

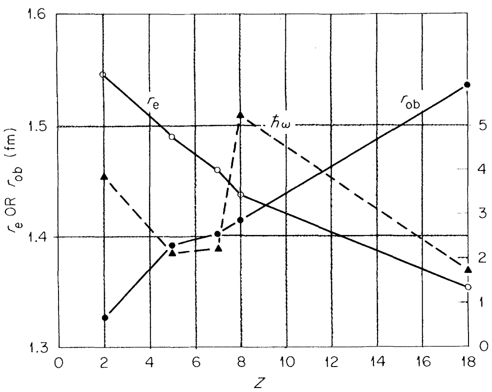

FIG. 2. The set of barrier parameters obtained in the least-squares fit to the total reaction cross sections. The effective radius parameter r_e is defined in terms of the barrier $\bar{E}_0$ by Eq. (21), and the barrier radius parameter r_{0b} is defined by $R_0 = r_{0b}(A_1^{1/3} + A_2^{1/3})$.

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Comparison of Sub-Coulomb Stripping and Analog-Resonance Results near Closed Shells*

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Sub-Coulomb (d,p) reactions have been studied for nuclei near closed shells. Reduced normalizations were extracted from the data and compared with those calculated for the analogs of the parent states using three analog-resonance theories. The R-matrix theory gives the best agreement with the (d,p) data.

In the study of isobaric analog resonances (IAR), spectroscopic factors have been extracted from proton elastic scattering by several different methods. These include the R-matrix approach of Thompson, Adams, and Robson[1] (TAR), and two shell-model methods, that of Mekjian and McDonald[2] (MM) and that of Zaidi and Darmodjo and Harney[3] (ZDH). These theories have been compared and their differences delineated on a theoretical level by Harney and Weidenmüller[4] (HW). Since these differences produce spectroscopic factors which may differ by as much as 50%, it seems desirable to attempt to determine which theory is the most nearly correct by using some experimental means independent of proton elastic scattering.

The method that has been employed in the past is to compare the spectroscopic factors S derived from (p,p_0) scattering via IAR with those found from (d,p) stripping to the low-lying parent states. Even though S is in principle model independent, in the distorted-wave Born-approximation (DWBA) analysis of the (d,p) data, S is strongly dependent on the optical-model parameters used. In many cases, the analysis of the (d,d) elastic-scattering data leads to several equally good families of parameters, which, when applied to the (d,p) reaction, yield spectroscopic factors which may dif-