P state since the identification of this state with ^{3}He-B has been suggested, for example, by Anderson and Brinkman.[17] On the other hand, Soda and Yamazaki[18] recently suggested F-state pairing for ^{3}He-A and D-state pairing for ^{3}He-B. The D- and BW P-state weak-coupling theoretical curves[19] are given on Fig. 2. Even at $T = 0$ the BW-state susceptibility is 0.35 of χ_n, so it appears, using weak-coupling theory, that ^{3}He-B is not a pure BW state. The experimental data are rather close to but apparently not coincident with the theoretical curve for D-state pairing. We note in this connection that Eq. (1) was derived for S-state pairing so that it may not be accurately applied to the present case although reasonable agreement is expected.[14] Further, the specific-heat ratio[1] at T_c is greater than that expected[20] for weak-coupling theories and is a weak function of pressure. Hence we might also expect a more complicated behavior for the susceptibility, rather than a universal, pressure-independent behavior scaling with T_c.

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Decay of Correlations

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For lattice systems with a symmetric transfer matrix, the correlation functions decay exponentially with distance if the fugacity z lies in a region of the complex plane that contains the origin and is free of zeros of the grand partition function. When these interactions decay slower than exponentially the correlations do not decay exponentially for small z and, for Ising ferromagnets with pair interactions, for all values of the magnetic field.

The correlations between widely separated regions of a thermodynamic (infinite) system are of great interest. They play a central role in the theory of critical phenomena[1] and help us to understand the microscopic structure of thermodynamic systems.[2] (They also play an important role in recent work in field theory.[3]) We are interested here in how the asymptotic decay of the correlation functions at large distances is related to the analyticity properties of the free energy as a function of the thermodynamic parameters of the system, in particular the fugacity z or magnetic field.

It is generally felt[1] that analyticity ought to imply exponential decay of the correlations, at least for finite-range potentials and nonordered, i.e., noncrystalline, systems. Indeed in many cases where analyticity can be proven, such as at low fugacities and/or high temperatures (or for fugacities corresponding to nonvanishing external magnetic fields in Ising-spin systems with ferromagnetic pair interactions), exponential decay can also be proven if the interaction has finite range.[2,4] The general relationship is, however, not proven, despite the fact that exponential decay is tacitly assumed to hold for one-phase systems in many discussions of critical indices. In this note we describe several results which bear on this question.

The first result deals with d-dimensional lattice systems (lattice gases or Ising spins) with translation-invariant pair interactions which are bounded from below. The range of the potential is such that, for a cubical lattice with unit spacing, the interaction between lattice sites m and n vanishes unless $|m_\alpha - n_\alpha| \leq 1$, $\alpha = 1, \ldots, d$, e.g., nearest- and next-nearest-neighbor interactions for $d=2$. Consider a system in a rectangular parallelepiped Λ. We impose "periodic boundary conditions" on Λ. Let D be a simply connected open region in the complex z plane which is free of zeros of the grand partition function $\Xi(z; \Lambda)$ for all sufficiently large Λ. We know from the Yang-Lee theorem[2] that the thermodynamic (infinite volume) free energy per site, $f(z)$, is analytic in the region D. (We treat the temperature as constant here and omit it from the notation.) The following theorem tells us about the decay of the correlations in D.

Theorem 1.—If D contains the origin, then for any real z in D, and in the thermodynamic limit, the correlation between any two sets of lattice sites decays exponentially as the two sets get further apart.

The results of Theorem 1 can be applied directly to various spin systems, both ferromagnetic and antiferromagnetic, and lattice systems with extended hard cores, for which regions D satisfying the conditions of Theorem 1 have been found by various authors.[5] We note in particular the following corollary.

Corollary to Theorem 1.—The Ising ferromagnet with pair interactions of range specified before has exponential decay (a) when the magnetic field $h \neq 0$ and (b) at $h=0$ and temperatures high enough for the point $h=0$ (i.e., $z=1$ in proper units) not to be a limit point of zeros of the grand partition function.

Part (a) of the Corollary is a known result[4]; part (b) is new and supports the devotees of critical-point exponents in their faith that the correlations decay exponentially for all temperatures above the critical temperature T_c.[1]

Theorem 1 is established by proving that a gap in the spectrum of the transfer matrix persists in the thermodynamic limit $K \to \infty$, where K represents a cross section of Λ perpendicular to the direction of transfer. (The existence of such a gap for finite K and real z is already known.) The theorem remains valid for higher-order Ising-spin systems (and their suitably defined continuum spin analogs[3]); cf. discussion at the end of this note.

Our second result shows that there is no exponential decay if the interaction potential falls off slower than any decaying exponential; for example, as some negative power of the distance.

Theorem 2.—For any lattice gas, Ising model, or particle system with two-body interactions whose potential has constant sign at large distances and does not decay exponentially (i.e., it decays more slowly than any decaying exponential), the infinite-volume correlation functions also do not decay exponentially for $|z| < a$, except possibly on a set whose intersection with any arc has zero arc-length measure. Here a is the lower bound on the radius of convergence of the Mayer z series, given by Ruelle.[2]

While Theorem 2 is restricted to small fugacities, the result probably holds for the whole gas phase. It therefore indicates that for Lennard-Jones–type potentials one should not assume exponential decay when defining critical exponents. Theorem 2 can also be extended to lattice systems with many-body interactions, for which the existence of a finite radius of convergence of the fugacity expansion has been proven by Gallavotti and Miracle-Sole.[6]

For ferromagnetic Ising models we can use the inequalities of Griffiths, Hurst, and Sherman[7] to extend the validity of Theorem 2. This leads to the following result.

Theorem 3.—For an Ising ferromagnet with two-body interactions whose potential does not decay exponentially, the two-body Ursell function does not decay exponentially for any real magnetic field, at any finite temperature.

The proofs of Theorems 1 and 2 depend on three lemmas, which we give below, and also on the properties of subharmonic functions.[8] A function v defined on an open connected region D on the plane, and capable of the value $-\infty$, as well as any real value, is said to be subharmonic if (i) for any open connected region E whose closure lies within D, the values of v inside E do not exceed those of the harmonic function taking the same values as v on the boundary of E, and (ii) v is upper semicontinuous and does not take the value $-\infty$ everywhere in D.

We note that every harmonic function is also subharmonic and that if φ is harmonic in D, the function $\ln|\varphi(z)| = \mathrm{Re}[\ln\varphi(z)]$ is subharmonic in D even if $\varphi(z)=0$ at some (but not all) points in D.

Lemma 1.—For a lattice system, if D is any connected open set in the complex plane containing part of the positive real axis and in which $\Xi(z;\Lambda)$ has no zeros for large Λ, then $\ln|\lambda_2(z;K)/\lambda_1(z;K)|$ is subharmonic in D for large K, where $\lambda_1, \lambda_2, \ldots$ are the eigenvalues of the transfer matrix for cross section K arranged in order of decreasing modulus (whether this defines them uniquely or not).

Lemma 2.—If $v_1, v_2, \ldots$ is a sequence of nonpositive subharmonic functions on a region (connected open set) D of the complex z plane, and if

$$\limsup_{n\to\infty} v_n(z_0) = 0$$

for some $z_0 \in D$, then for any differentiable arc A of finite length within D we have

$$\limsup_{n\to\infty} v_n(z) = 0$$

for almost all z in A, where "almost all" refers to the measure based on arc lengths of A.

Lemma 3.—For an Ising model or lattice gas, when $z>0$, we have

$$|\lambda_2(z;K)/\lambda_1(z;K)| \le |z/a|,$$

where a is defined in Theorem 2.

To prove Theorem 1 we first use Lemma 1, which tells us that for each K the function $v(z;K) = \ln|\lambda_2(z;K)/\lambda_1(z;K)|$ is a subharmonic function of x, y for $z=(x+iy)\in D$. It is also, by definition, nonpositive. Lemma 2 shows that if, for an increasing sequence of cross sections K, the sequence $v(z_0;K)$ were to take arbitrarily small negative values for some z_0 in D, then it would have to do so for almost all z_0 in D. We know, however, from Lemma 3 that $v(z;K)$ is bounded away from zero for small values of $|z|$; hence, it is bounded away from zero for any z in D. That is, the gap at the top of the spectrum of the transfer matrix does not tend to 0 for large K. Theorem 1 then follows by considering the expressions[9] giving the correlation functions in terms of eigenvalues and eigenvectors of the transfer matrix.

Theorem 2 is proven by considering subharmonic functions such as $v(z;\vec{r}) = (|\vec{r}|+1)^{-1}\ln|u_2(z;\vec{r})/Mz^2|$, where $u_2(z;\vec{r})$ is the infinite-volume two-particle Ursell function, which has a bound of the form $|u_2(z;\vec{r})| < M|z|^2$ for $|z|<a$. At $z=0$ we have $v(0;\vec{r})\to 0$ as $|\vec{r}|\to\infty$, and by Lemma 2 we must then have $v(z;\vec{r})\to 0$ for almost all z in R. The proof of Theorem 3 then follows from Theorem 2, as outlined earlier.

The proof of Lemma 1 uses methods developed by Penrose and Elvey.[10] It is shown that $\ln|\lambda_1(z;K)|$ is harmonic in D and that $\ln|\lambda_2(z;K)|$ is the maximum of a finite family of harmonic functions and is therefore subharmonic. The proof of Lemma 2 follows essentially from the definition of subharmonic functions. Lemma 3 is proved from the fact that the correlations in the region of convergence of the fugacity expansion decay exponentially.[2,4,11]

Details of the proofs and more general forms of the results will be published elsewhere. We wish to note here, however, that it is only in the connection between the gap in the transfer matrix and the decay of the correlations that the severe restrictions on the range of the interactions (which make the transfer matrix symmetric) are used. The persistence of the gap as $K\to\infty$, at all points z of a region E free of zeros of the grand partition function and containing an arc on which the gap is known to persist, holds for more general finite-range interactions and remains valid also for continuum fluids with hard cores (or other sufficiently strong repulsions at close approach). Also, the fugacity z can be replaced by one or more other parameters in the potential for some ranges of which one has information about the persistence of the gap in the transfer matrix. Finally, for Ising ferromagnets the Corollary to Theorem 1 holds for pair interactions with arbitrary finite range. This can be proven by combining the ideas of this note with results obtained in Ref. 4.

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Enhanced Microwave Emission Due to the Transverse Energy of a Relativistic Electron Beam

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The role of the transverse energy of a magnetically focused intense relativistic electron beam in the emission of microwaves is investigated experimentally and theoretically.

There has been considerable interest recently in the production of high-power microwaves by the pulsed intense relativistic electron beams which have become available in the last few years.[1,2] Experimental observations of high-power microwaves have been made in a variety of beam configurations, including beams injected into a few hundred milliTorr of neutral gas,[3] and magnetically focused annular beams propagating in vacuum ($<10^{-3}$ Torr) in the presence of special metal boundaries[4,5] and magnetic field perturbations.[6] Theoretical models have been developed, and calculations have been carried out to try to explain the observations,[7-10] but they have been hampered by an incomplete knowledge of the properties of the electron beam. In the present study, we control the transverse energy of beam electrons by varying the static spatial magnetic compression to which the beam, propagating in a straight, metallic drift-tube wave guide, is subjected. We find that if this transverse energy exceeds a certain minimum value, microwave power in excess of that possible by a single-particle mechanism is obtained. Moreover, we find this to be a characteristic of microwave production using the perturbed magnetic-field configuration. A theoretical model based on an interaction between the observed wave-guide mode and the electron beam gives unstable waves which agree well with observations.

The experimental configuration used for the present study is shown schematically in Fig. 1. The electron beam is produced by applying a high-voltage pulse from a 7-Ω, 50-nsec pulse-forming line to a foilless diode.[11] The diode voltage and current are 350–650 kV and 10–25 kA, respectively. The beam propagates in a 4.7-cm-i.d., thin-walled, stainless-steel drift tube immersed in a quasistatic (10-msec risetime) magnetic field applied coaxially to the drift tube by a 22-cm-diam, 1-m-long solenoid. By varying the distance d in Fig. 1 between the cathode and the end of the solenoid from -2 cm (i.e., cathode 2 cm inside the coil) to 18 cm, the magnetic field near the cathode relative to that in the middle of the coil is varied from ~ 0.55 to ~ 0.1. Thus, the distance d controls the magnitude of the radial component of the magnetic field near the cathode. The interaction of the beam electrons with the radial field produces the desired transverse energy. Lucite witness plates obtained in the middle of the solenoid for $d=-2$ and 1 cm are shown on the right-