tions, and the parameters are not readily identifiable with simple qualities of the potential. They nevertheless indicate a Π-state well with a depth of about 2000 cm^{-1}, the repulsive wall of which begins to have negative curvature above about 1000 cm^{-1} above the top of the well.

Inertial-Range Spectrum of Turbulence*

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We argue that the value of μ in Kolmogorov's energy spectrum formula $E(k) \propto k^{-5/3-\mu}$ cannot be deduced from general principles. The argument is supported by exhibiting a modified Navier-Stokes equation which has the same dimensionality, symmetries, invariances, and equilibrium statistical ensembles as the original but gives a drastically different inertial range.

Kolmogorov's formula for the inertial-range spectrum of high-Reynolds-number, incompressible, three-dimensional, Navier-Stokes (NS) turbulence is

$$E(k) = C\epsilon^{2/3} k^{-5/3} (kL)^{-\mu}. \qquad (1)$$

Here $E(k)$ is the wave-number spectrum of kinetic energy, C is a dimensionless parameter of order 1, ϵ is the rate of energy dissipation by viscosity, per unit mass, and μ is a parameter which is zero in Kolmogorov's 1941 theory[1] and >0 in the modified theory of 1962.[2,3] Equation (1) applies to the inertial range of wave numbers $L^{-1} \ll k \ll k_d$, where L is the macroscale, at which energy is fed into the turbulence, and $k_d = (\epsilon/\nu^3)^{1/4}$ (ν is the kinematic viscosity) is the approximate wave number where dissipation becomes strong.

The 1941 theory assumes that the inertial-range wave numbers exhibit an energy cascade, from low wave numbers to high, which is (a) local in wave number; (b) characterized by self-similar statistical distributions at all inertial-range scales; and (c) dependent on the macroscale statistics only through ϵ. The 1962 theory invokes a modified picture in which there is a self-similar increase in spatial intermittency of the velocity differences $\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})$ ($|\vec{r}| \sim 1/k$) at each cascade step, resulting in increasing efficiency of energy cascade as k rises.[4-6] The parameter μ also appears in other, related predictions of the 1962 theory. The latter is at least approximately supported by a variety of geophysical experiments, which consistently yield $\mu \sim 0.05$.[7-9] Measurements of $E(k)$ itself at high Reynolds numbers are consistent with (1), but cannot distinguish $\mu = 0$ from $\mu \sim 0.05$.[10,11]

Equation (1) has not been derived from the NS equation in any solid way, and the difficulties in trying are severe and well known.[6] This situation makes attractive some recent speculations by Martin[12] and Nelkin[13] that (1) can be validated, and μ determined, from general statistical-mechanical principles, based on dimensionality, invariances, and symmetries, without the need of calculations involving the detailed structure of the NS equation. In particular, Martin and Nelkin point out the possibility of an analogy between μ and the universal exponents of critical-point phenomena.

Such suggestions should be pursued. But we think it timely to make a counterargument of fundamental nature, based on the fact that the inertial-range cascade is a state of strong departure from absolute statistical equilibrium and thereby differs qualitatively from states of thermal fluctuation about absolute equilibrium.

The NS equation in a cyclic box of side $\gg L$ may be written

$$(\partial/\partial t - \nu\nabla^2) u_i(\vec{x}) = -P_{ij}(\nabla)\{[\vec{u}(\vec{x})\cdot\nabla] u_j(\vec{x})\}, \qquad (2)$$

where $P_{ij}(\nabla)$ is the solenoid (transverse) projection operator, defined by

$$P_{ij}(\nabla) = \delta_{ij} - \nabla^{-2}\partial^2/\partial x_i \partial x_j,$$
$$\nabla^{-2} f(\vec{x}) \equiv -(4\pi)^{-1} \int f(\vec{x}')\,|\vec{x}-\vec{x}'|^{-1} d^3x'. \qquad (3)$$

Under the transformation $\vec{u}(\vec{x}) = \sum \vec{h}(\vec{k}) \exp(i\vec{k}\cdot\vec{x})$, (2) becomes

$$(\partial/\partial t + \nu k^2) h_i(\vec{k}) = -i(\delta_{ij} - k_i k_j/k^2) \sum_{\mathbf{p}} h(\vec{k}-\vec{p})\cdot\vec{p}\, h_j(\vec{p}), \qquad (4)$$

where the sums are over all allowed wave numbers.

If $\nu=0$ and (4) is consistently truncated to $k<K$ by removing all terms where p or $|\vec{k}-\vec{p}|$ exceeds a cutoff K, the resulting system obeys Liouville's theorem, conserves the energy $\sum|\vec{u}(\vec{k})|^2/2$, and, consequently, has absolute equilibrium ensembles of the form $E(k)\propto k^2$.[14] Since the effective range of energy-transferring dynamical interaction in the wave-number space is a wave-number ratio of order 2 or 4, the departure from absolute equilibrium represented by (1) is comparable to that in a gas of particles where the temperature changes by its own order of magnitude in one mean free path.[6]

The inertial range must therefore be viewed as a transport phenomenon in k space, rather than a state of fluctuation about equilibrium. As such, it is to be expected that μ should be influenced by two factors which depend on the detailed structure of (2): the effective cascade step size (typical ratio of energy-exchanging wave numbers) and the effective statistical spread introduced at each eddy breakdown. These factors together determine how fast spatial intermittency increases with decrease in scale size.

The dependence of (1) on details of structure is supported by consideration of generalized NS equations of the form

$$(\partial/\partial t-\nu\nabla^2)u_i(\vec{x})=-P_{ij}(\nabla)[\vec{v}(\vec{x})\cdot\nabla u_i(\vec{x})], \qquad (5)$$

where $\vec{v}(\vec{x},t)$ is any solenoidal functional of $\vec{u}(\vec{x},t)$ which satisfies $\int\vec{v}(\vec{x})\,d^3x=\int\vec{u}(\vec{x})\,d^3x$, where the integrals are over the cyclic box. With these conditions, (5) gives conservation of $\int|\vec{u}(\vec{x})|^2d^3x$ by the right-hand side, exhibits Galilean invariance,[6] and has the same inviscid-equilibrium equipartition distribution as the original NS equation.

The inertial-range cascade properties depend very much on what $\vec{v}(\vec{x})$ is. Consider

$$\vec{v}(\vec{x})=\exp(L^2\nabla^2)\vec{u}(\vec{x}), \qquad (6)$$

where L is an intrinsic length. The operator $\exp(L^2\nabla^2)$ is nonlocal in $\vec{x}$ space, but so already is $P_{ij}(\nabla)$. Let a statistically steady state be maintained by driving the system at wave numbers $\sim 1/L$ with a forcing term on the right-hand side of (5). High wave numbers in the $\vec{v}$ field are suppressed by $\exp(L^2\nabla^2)$, with the result that the effective straining field acting on u-field scales $\ll L$ is confined to the input wave numbers. Consequently, the basis for the completely local cascade in k space which underlies (1) is entirely destroyed.

Instead, individual small-scale $\vec{u}$-field structures have negligible reaction back on the straining field, and the inertial-range energy transport is effectively a linear process, analogous to the straining of a passive scalar field by $\vec{v}$. Then the arguments of Batchelor[15] imply that the energy transport rate is proportional both to $E(k)$ and to the typical straining rate $\eta\sim v_L/L$, where v_L is the root-mean-square velocity in wave numbers $\sim 1/L$. By either dimensional analysis or detailed derivation like that of Ref. 15, the result is

$$E(k)\sim(\epsilon/\eta)k^{-1} \quad (kL\gg 1). \qquad (7)$$

Equation (7) departs drastically from (1), which is experimentally supported, with $\mu\ll 1$, for the NS equation. This qualitative difference in nonequilibrium transport behavior exists despite the fact that the modified equation has the same essential invariances, symmetries, and dimensionality as the NS equation. The inviscid absolute equilibrium distributions are identical in the two cases, if similar K cutoffs are made.

The physics of repeated random straining[16] imply that spatial intermittency of the $\vec{u}$ field increases with decrease of scale size in the cascade described by (7). Because of the effective linearity, this does not affect the exponent in (7).

How fundamental theoretical attack on the inertial-range problem should proceed is unclear. No *formal* analysis by perturbation theory, the moment-equation hierarchy,[17] or renormalization techniques[6] can settle whether (1) is a valid equation and, if so, whether $\mu=0$ or not. Orszag[17] has shown that $\mu=0$ is formally consistent with every order of the moment hierarchy and, indeed, arises from the hierarchy under the assumption that the moments and the cumulants of any given order all go as the same power of k. But if this assumption is not made, then $\mu\neq 0$ also is formally consistent. The Eulerian renormalized perturbation series for the velocity covariance formally yields $E(k)\propto k^{-3/2}$, to every order, while low-order closure schemes that are invariant to random Galilean transformations lead to (1) with $\mu=0$.[6] Such schemes cannot embody the higher statistics associated with intermittency buildup. The real questions concern which, if any, of the formal solutions imply everywhere non-negative probability distributions and thereby satisfy all realizability inequalities. From a physical point of view the choice between $\mu=0$ and $\mu\neq 0$ in (1) is by no means obvious *a priori*. The idea of a chain of stochastic eddy breakdowns leads naturally to increasing intermittency along the chain.[4-6] But the NS equation also includes interactions among similar scales, corresponding to spatial mixing of energy and, hence, cross-linking of the cascade chains. The question of whether such cross-linking is strong enough to limit intermittency buildup needs an answer.[18]

One thing feasible is the exploration of model systems [(5) and (6) are an example] which are more transparent than the NS equation and which may lead to insights. General models of conservative cascade chains, of the forms

$$dy_n/dt = \sum_m A_{nm} y_m \quad (A_{nm} + A_{mn} = 0) \tag{8}$$

and

$$dy_n/dt = \sum_{ml} A_{nml} y_m y_l \quad (A_{nml} = A_{nlm},\ A_{nml} + A_{mln} + A_{lnm} = 0), \tag{9}$$

can be explored simultaneously by analysis and by computer simulation.[19] The following caution must be observed in relating models with limited numbers of y's to the NS equation. Even when there is extreme spatial intermittency at small scales, the univariate distributions of the individual Fourier amplitudes in infinite, homogeneous turbulence with finite correlation scales are *accurately normal,* by the central limit theorem, solely as a consequence of homogeneity.[20] Spatial intermittency is a collective phenomenon in the Fourier representation.

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Static Nuclear Magnetism in Extraordinary Liquid ^{3}He†

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Static-nuclear-magnetization measurements show temperature-independent magnetization in the "A" phase of ^{3}He, temperature-dependent magnetization in the "B" phase of ^{3}He, and at the boundary between these phases a discontinuity in magnetization which approaches zero at a polycritical point.

In this Letter we present the first measurements of static nuclear magnetism in an all-liquid sample of ^{3}He. Below the line of second-order transitions[1] at T_c the P-T phase diagram is split into two parts by a line T_{AB} of magnetization discontinuities extending from the melting curve at a temperature probably that of "B"[2] to the T_c line at a pressure of 21.7 bar. The magnetization discontinuity approaches zero as the T_{AB} line approaches the T_c line so the measurements