tions from spherical symmetry. We will sketch the derivation of this result here; the details will be published elsewhere.

Choose the apex, P, of the unperturbed null cone so that it lies at the center of spherical symmetry and is sufficiently close to the singularity so that the past null cone intersects no matter outside the Schwarzschild event horizon. Outside the horizon, then, Φ_{11} and Λ can be assumed always to vanish. The affine parameter r can be chosen so that the two-surfaces of constant r are natural spheres left invariant under the rotations which define the spherical symmetry. The quantity τ vanishes for the unperturbed spherical collapse and, therefore, the first-order perturbations in it determine $\dot{A}$ to *second order*. The perturbation in τ can be found by writing out the perturbed Newman-Penrose equations and separating the angular dependence by expanding in spin-weighted spherical harmonics as in Ref. 11. Letting $\tau^{(1)}$ denote the radial part of the perturbation in τ corresponding to a particular multipole l, one finds for the physically interesting $l \geq 2$ cases

$$\bar{\tau}^{(1)}(r) = Kr^{-2} + [(l-1)(l+2)/2]^{1/2} r^{-2} \times \int_\infty^r (dz/z^2) \int_\infty^z dx\, x^3 \Psi_4^{(1)}(x), \quad (8)$$

where the integrals are taken in the past null cone of P, $\Psi_4^{(1)}$ is the radial part of the perturbation of the Riemann tensor component $-R_{\alpha\beta\gamma\delta} \times n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$, and K is a constant depending on how r is chosen. The quantity $\Psi_4^{(1)}$ itself is the solution of a second-order homogeneous linear differential equation.[11] The solution corresponding to ingoing waves on the horizon is bounded there. Sufficiently far in the past, $\Psi_4^{(1)}$ will be calculable from the static precollapse geometry and, for the physically interesting $l \geq 2$ perturbations, will fall off sufficiently fast to make the integrals in Eq. (8) converge. The perturbation $\tau^{(1)}$ therefore will be finite in the neighborhood of the unperturbed horizon. Since the size of the perturbation is by definition small, one concludes by integrating Eq. (1) that an average trapped sphere must be formed slightly inside the unperturbed horizon to second order in the deviation from spherical symmetry. In other words, the property of a spherical collapse that average trapped surfaces are formed is stable under second-order perturbations from spherical symmetry.

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Proof of the Pomeranchuk Theorem for Unbounded Total Cross Sections

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If particle or antiparticle total cross sections σ_P, σ_A are unbounded as $E \to \infty$, using unitarity it is shown that on the average the ratio $\langle \sigma_P - \sigma_A \rangle / \langle \sigma_P + \sigma_A \rangle \to 0$.

Recent experimental data at the CERN insecting storage rings[1] indicate that the proton-proton total cross section increases with laboratory energy beyond the 300-GeV region. At the equivalent laboratory energy of 1500 GeV the proton-proton total cross section reaches 43 mb. The antiproton-proton total cross section decreases steadily as energy increases and reaches the same value at 60 GeV, the highest energy where experimental data are available. It is therefore of considerable interest to find out whether the Pomeranchuk theorem in the form $(\sigma_P - \sigma_A)/(\sigma_P + \sigma_A) \to 0$ can be violated, where σ_P and σ_A are particle and antiparticle total cross sections. The purpose of this note is to give a proof that this is not possible, if σ_P or σ_A are unbounded.

Using unitarity, Eden and Kinoshita[2] proved that, if as $E\to\infty$ it is true that $\sigma_P \sim C_P(\ln E)^m$ and $\sigma_A \sim C_A(\ln E)^m$, then $C_P = C_A$. Because of the special form assumed for the behavior of the total cross sections, their demonstration is not sufficiently general. It is desirable to have a general proof using only consequences of axiomatic field theory. A step forward in this direction was recently made by Truong and Lam.[3] They proved that if there was no cancelation between the zeroth and the first moment of $\Delta\sigma$, then

$$\lim_{E\to\infty} \frac{\langle\Delta\sigma\rangle_E}{\langle\sigma_P\rangle_E + \langle\sigma_A\rangle_E} = 0, \tag{1}$$

where $\ln E \langle\Delta\sigma\rangle_E = \int_0^E dE'\, \Delta\sigma(E')/E'$ with similar definitions for $\langle\sigma_P\rangle_E$ and $\langle\sigma_A\rangle_E$. Although it is difficult to construct simple functions where cancelation exists, their proof cannot be regarded as completely satisfactory. We give here a different method to prove rigorously Eq. (1). The problem of oscillations of the total cross sections is controlled by the averaging procedure as used in the Ref. 3 and also by the properties of the univalent functions introduced earlier by Khuri and Kinoshita.[4]

For simplicity we assume the existence of a dispersion relation. The following proof is also valid if there is a finite region of nonanalyticity. Let us denote by $f_{P,A}(E)$ the forward particle and antiparticle amplitudes with E the laboratory energy. The optical theorem is $\mathrm{Im} f(E) = (4\pi)^{-1} q \times \sigma_{\mathrm{tot}}(E)$, where q is the laboratory momentum. We begin first by writing the inverse dispersion relation for the forward amplitude $f_a \equiv f_P - f_A$:

$$\mathrm{Im} f_a(E) = -\frac{2E^2 p}{\pi} \int_0^\infty \frac{[\mathrm{Re} f_a(E') - bE']\,dE'}{(E'^2 - E^2)E'}, \tag{2}$$

where b is a subtraction constant at $E=0$. Let us integrate Eq. (2):

$$\int_0^E \mathrm{Im} f_a(E') \frac{dE'}{E'^2} = -\frac{1}{\pi}\int_0^\infty \ln\left|\frac{E'+E}{E'-E}\right| \mathrm{Re} f_a(E') \frac{dE'}{E'^2}, \tag{3}$$

where we have interchanged orders of integration and neglected a constant on the right-hand side (rhs) of (3). Since $\mathrm{Im} f(E) = 0$ for $|E| < m$, the integral converges at $E=0$. This equation is important to derive asymptotic theorems described below.

Let us now improve the Froissart-Martin[5] bound for $\langle\Delta\sigma\rangle_E$. Since $|f_a(E)| \le CE \ln^2(E)$ for E sufficiently large, where $C = 2/t_0$, with $t_0 = 4m^2$ and m the pion mass, then using (3), we have for E sufficiently large

$$\left|\int_0^E \mathrm{Im} f_a(E') \frac{dE'}{E'^2}\right| < \frac{C}{\pi}\int_0^\infty \ln\left|\frac{E'+E}{E'-E}\right| \ln^2 E' \frac{dE'}{E'^2}. \tag{4}$$

After changing the variable of integration to $x = E'/E$, performing the integration, and using the optical theorem and the definition of $\langle\Delta\sigma\rangle_E$ we have

$$\langle\Delta\sigma\rangle_E \le (4\pi^2/t_0) \ln E. \tag{5}$$

This result improves the Froissart-Martin bound by a power of $\ln E$.[6] It should be emphasized that no assumption was made on the high-energy behavior of $\Delta\sigma$ to derive (5). From this bound, it is clear that if the sum $\sigma_P + \sigma_A$ increases faster than $\ln E$, then the Pomeranchuk theorem in the form (1) is proved. More precisely, if

$$\lim_{E\to\infty} \frac{\langle\sigma_P\rangle_E + \langle\sigma_A\rangle_E}{\ln E} = \infty$$

then (5) implies (1). The proof given below is valid for a more general situation.

We note parenthetically that by using the same method, if $|f_a(E)|/E \ln E \to 0$, the usual Pomeranchuk theorem for bounded total cross sections can be proved in a simple manner, i.e., $\langle\Delta\sigma\rangle_E = 0$ as $E\to\infty$ as derived by Truong and Lam[3] from the proof of Martin.[7] The Pomeranchuk theorem for the differential cross section can also be proved by an equation similar to (3).[8]

We now give a general proof of (1) using analyticity and unitarity in the form[2,6]

$$|f_{P,A}| \leq EC'(\ln E)\sigma_{P,A}^{1/2},$$

where $C' = 1/2(\pi t_0)^{1/2}$. ($\sigma_{P,A}$ can also be replaced by the corresponding elastic cross sections.) Using these bounds in (3) we have

$$\left|\int_0^E \mathrm{Im} f_a(E')\frac{dE'}{E'^2}\right| \leq \frac{C'}{\pi}\int_0^\infty \ln\left|\frac{E'+E}{E'-E}\right| \ln E'[\sigma_P^{1/2}(E') + \sigma_A^{1/2}(E')]\frac{dE'}{E'}. \tag{6}$$

Our purpose is to express the rhs of (6) in terms of quantities which are easier to handle. To do this, let us construct the following function $H(E)$ which is analytic in the upper half plane:

$$H(E) = \frac{2E}{\pi}\int_0^\infty \frac{\ln E'[\sigma_P^{1/2}(E') + \sigma_A^{1/2}(E')]}{E'^2 - E^2}dE'. \tag{7}$$

Following Khuri and Kinoshita[4] let us construct the (univalent) function $G(E)$ which is also analytic in the upper half E plane:

$$G(E) = \int_0^E H(E')dE'/E' \tag{8a}$$

whose real part, apart from the factor C', is the rhs of Eq. (6), and whose imaginary part is

$$\mathrm{Im}G(E) = \int_0^E \ln E'[\sigma_P^{1/2}(E') + \sigma_A^{1/2}(E')]dE'/E'. \tag{8b}$$

From the Froissart-Martin bound for $\sigma_{P,A}$ we have $0 < \mathrm{Re}G(E) < \ln^2 E$ and $0 < \mathrm{const} < \mathrm{Im}G(E) \leq \ln^3 E$. From these bounds, it is clear that $G(E)$ is a slowly varying function, and since it is antisymmetric under crossing, we expect $(\mathrm{Re}G/\mathrm{Im}G) < C/\ln E$. In fact Khuri and Kinoshita[4] show that, in order for the upper bound for $|G(E)|$ to be valid, there must be at least an infinite sequence of points $\{E_i\}$, $E_i \to \infty$ as $i \to \infty$, such that

$$\frac{\mathrm{Re}G(E_i)}{\mathrm{Im}G(E_i)} < \frac{\mathrm{const}}{\ln E_i}. \tag{9}$$

Using this in (6) together with (8b) and the method to be described below, we can show that there exists an infinite sequence of points $\{E_i\}$, $E_i \to \infty$, such that

$$\langle\Delta\sigma\rangle_{E_i}/\langle\sigma_P + \sigma_A\rangle_{E_i} \to 0, \tag{10}$$

which is the desired result. This can be made more precise as follows.

It can be shown that by using the phase representation of $G(E)$ the following equation is valid:

$$\frac{1}{N(a)}\int_{E(1-a)^{1/2}}^{E(1+a)^{1/2}} \ln|G(E')|\frac{dE'}{E'} = \frac{2}{\pi}\int_0^E \frac{\Delta(E')}{E'}dE' + O(1), \tag{11a}$$

where $\Delta(E) = \pi/2 - \delta(E) > 0$, $\delta(E)$ is the phase of $G(E)$, $N(a) = \frac{1}{2}\ln|(1+a)/(1-a)|$, and $0 < a < 1$. From the bounds of $|G(E)|$ it follows that

$$(2/\pi)\int_0^E \Delta(E')dE'/E' \leq 3\ln\ln E. \tag{11b}$$

Inequality (11) shows that $\Delta(E) < \frac{3}{2}\pi/\ln E$ except on a set of points which has an asymptotic zero density. Since $\Delta(E) \to 0$, we can replace it by $\mathrm{Re}G/\mathrm{Im}G$. Hence

$$\mathrm{Re}G(E)/\mathrm{Im}G(E) < \tfrac{3}{2}\pi/\ln E. \tag{12}$$

Using inequality (12) in (6), we have

$$|\langle\Delta\sigma\rangle_E| \leq \frac{6\pi^2 C'}{\ln^2 E}\int_0^E \ln E'[\sigma_P^{1/2}(E') + \sigma_A^{1/2}(E')]\frac{dE'}{E'}. \tag{13}$$

Using the Schwartz inequality for the rhs of (13), we arrive at

$$|\langle\Delta\sigma\rangle_E| \leq \left(\frac{3}{2}\right)\frac{2\pi^2}{(3\pi t_0)^{1/2}}[(\langle\sigma_P\rangle_E)^{1/2} + (\langle\sigma_A\rangle_E)^{1/2}]. \tag{14}$$

Dividing both sides of (14) by $\langle\sigma_P+\sigma_A\rangle_E$, it is straight forward to show

$$\frac{\langle\Delta\sigma\rangle_E}{\langle\sigma_P+\sigma_A\rangle_E}\leqslant\frac{\sqrt{3}\pi^{3/2}}{\sqrt{t_0}}\left[(\langle\sigma_P\rangle_E)^{-1/2}+(\langle\sigma_A\rangle_E)^{-1/2}\right]. \quad (15)$$

We define the unboundedness of the total cross sections as

$$\lim_{E\to\infty}\langle\sigma_{P,A}\rangle_E=\infty.$$

Hence if $\langle\sigma_P\rangle$ and $\langle\sigma_A\rangle$ are both unbounded the rhs of (15) is zero which is the desired result [Eq. (1)]. Equation (14) shows that the same result holds if only one of the total cross sections is assumed to be unbounded; it then follows that the other cross section must be unbounded, too. From Eq. (1) or (10) it is also clear that there exists a set of points $\{E_i\}$, $E_i\to\infty$, on which $\Delta\sigma/(\sigma_P+\sigma_A)\to 0$; in particular, if $\Delta\sigma/(\sigma_P+\sigma_A)$ has a limit, this limit is zero.

We should like to emphasize that inequality (14) is quite general; it is also valid if we replace $\langle\sigma_P\rangle_E$ and $\langle\sigma_A\rangle_E$ by the corresponding elastic cross sections. If one is willing to make a strong assumption that σ_P, σ_A are bounded then the coefficient $\frac{3}{2}$ on the rhs of (12) and (14) should be replaced by unity. In this case one gets a generalization the results of Roy and Singh.[6] For pion-nucleon scattering, if isospin invariance is assumed, a restriction on $\langle\Delta\sigma\rangle_E$ in terms of the charge-exchange cross section $\pi^-p\to\pi^0n$ can be similarly obtained.

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Neutrino-Induced Muons Deep Underground*

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We report here the results of a measurement of the integral energy distribution of the neutrino-induced muon flux deep underground using the Utah neutrino detector. Five neutrino-induced muon candidates have been obtained from the approximately 10^6 muons which have passed through the detector during 603 days of live operation out of 830 days total elapsed time. By comparing the observed and expected energy distributions we are able to set lower limits on the saturation energy E_0 of the total neutrino-nucleon cross section or, equivalently, on the mass M_W of the intermediate vector boson if scale invariance of the inelastic structure function νW_2 is assumed. We find at the 1σ, 2σ, and 3σ levels of confidence $E_0>320$, 80, and 29 GeV or $M_W>10$, 5, and 3 GeV, respectively.

The observation of a muon flux induced by the interaction of cosmic-ray neutrinos in rock deep underground has been used to investigate the nature of the weak interaction.[1-4] Previous experiments have yielded values for the integral muon flux, but none thus far has analyzed the high-energy spectrum of the observed muons. In this communication, we report the results of a measurement of the spectrum of the underground neutrino-induced muon flux made possible by the unique University of Utah neutrino detector.[5-8] We compare the observed spectrum with those calculated for different assumptions concerning the high-energy ($E_\nu>10$ GeV) behavior of the neutrino-nucleon total cross section.[9] Comments are made on the scale invariance of the structure functions measured in inelastic neutrino-nucleon scattering, and a lower limit is placed on the