tempted to use negative ions, but find the signal is smaller by nearly 2 orders of magnitude. While this is not completely understood, we believe that the reason is mainly that negative ions are more tightly bound to rings, and thus it is harder to form free ions from charged rings.

[8]M. Steingart and W. I. Glaberson, Phys. Lett. 35A, 311 (1971). Typical drag losses are a few volts (~ 4 V for data shown in Fig. 2) which is small compared to the apparent energy of the neutral vortices emitted from the microphone. This small energy loss could also explain why the time-of-flight deduced radii are slightly smaller than the cross-section deduced radii.

[9]Such a distribution in velocities could also be obtained with different quantum numbers; however, in such a case we would expect to see several peaks instead of a single smooth bell-shaped trace.

[10]The energy measurements, which will be published elsewhere, were obtained recently in another apparatus and they agree with the size results.

[11]G. Gamota and T. M. Sanders, Jr., Phys. Rev. Lett. 21, 200 (1968).

[12]The equations relating energy, velocity and radius are $\mathcal{E} = \frac{1}{2}\kappa^2 R\{\ln(8R/a) - 2\}$; $v = (\kappa/4\pi R)[\ln(8R/a) - \frac{1}{2}]$, where $\mathcal{E}$ is the energy, κ is the circulation, R is the ring radius, a is the core radius, and v is the velocity.

[13]G. Gamota, A. Hasegawa, and C. M. Varma [Phys. Rev. Lett. 26, 960 (1971)] show that the final width of a charged pulse of vortex rings varies inversely as the square root of the density.

Parametric Decay of Obliquely Incident Radiation*

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An electromagnetic wave obliquely incident on an inhomogeneous plasma has a resonance near $\omega_0 = \omega_{pe}(x)$. It is shown that temporally growing modes exist for parametric decay into ion-acoustic and electron-plasma waves in the vicinity of this resonance in spite of the density gradient. For typical laser fusion parameters the maximum growth rate is comparable to the ion-acoustic frequency. The mechanism can contribute to significant power absorption.

An electromagnetic wave of frequency ω_0 incident obliquely on an inhomogeneous plasma and with finite electric field component E_x parallel to the density gradient has a resonance near $\omega_0 = \omega_p(x)$, where great enhancement of E_x occurs. At this resonance, significant absorption takes place. In a cold plasma, this absorption is due to the breaking of the large-amplitude plasma wave, resonantly driven by E_x of the pump, as has been demonstrated by a recent numerical simulation.[1] In a hot inhomogeneous plasma, the plasma wave excited at resonance propagates into the low-density region and becomes Landau damped. For sufficiently large pump fields, parametric processes may invalidate the conclusions of linear wave-transformation theory.

Here, we consider the parametric decay of the pump field in the vicinity of the resonance into an electron-plasma and an ion-acoustic wave. Because of the resonant structure of the pump wave, temporally growing modes exist even in an inhomogeneous plasma. The growth rate of the most rapidly growing mode is, for typical laser fusion parameters, comparable to the ion-acoustic frequency. The decay waves are preferably aligned with the density gradient. The energy absorption coefficient has a maximum value of $\frac{1}{2}$.

Consider an obliquely incident electromagnetic wave on a plasma with a monotonically increasing density. Neglecting ion temperature and separating the high- and low-frequency response of the electrons, we find two coupled differential equations for the ion and electron density fluctuations[2]:

$$\left(\frac{\partial^2}{\partial t^2} + 2\nu_e\frac{\partial}{\partial t} + \omega_e^2 - v_e^2\nabla^2\right)n_e = \frac{e}{M}(\nabla n_i)\cdot\vec{E}_0, \quad \left(\frac{\partial^2}{\partial t^2} + 2\nu_i\frac{\partial}{\partial t} - \omega_i^2\lambda_D^2\nabla^2\right)n_i = \frac{e}{M}(\nabla n_E)\cdot\vec{E}_0, \tag{1}$$

where ω_e and ω_i are the respective electron and ion plasma frequencies, and v_e is the electron thermal speed. We have neglected terms of order $(\nabla E_0/E_0)(\nabla n_{e,i}/n_{e,i})^{-1}$.[3] Writing

$$n_e = a_1\exp\{i[\omega_1 t - k_1 x - \textstyle\int_0^x \Delta k_1(x')\,dx']\} + \text{c.c.}, \quad n_i = a_2\exp\{i[\omega_2 t - k_2 x - \textstyle\int_0^x \Delta k_2(x')\,dx']\} + \text{c.c.}, \tag{2}$$

where $\omega_1 = \omega_e(1 + 3k^2\lambda_D^2)^{1/2}$ and $\omega_2 = \omega_i k\lambda_D$ are the decay wave frequencies with $\omega_0 + \omega_1(0) + \omega_2 = \delta$, we then find, upon keeping only first-order variations of $a_i(x,t)$ in space and assuming that the time variation of a_i is slow compared to ω_1 but possibly comparable to ω_2, that

$$\left(\frac{\partial}{\partial t}+\nu_e-v_1\frac{\partial}{\partial x}\right)a_1=\frac{e}{2m\omega_1}a_2{}^*\vec{k}\cdot\vec{E}_0{}^*\exp[+i\int_0^x\kappa\,dx'-i\delta t],$$
$$\left[(2i\omega_2)^{-1}\left(\frac{\partial^2}{\partial t^2}+2i\omega_2\frac{\partial}{\partial t}\right)+\nu_i+v_2\frac{\partial}{\partial x}\right]a_2=\frac{e}{2M\omega_2}a_1{}^*\vec{k}\cdot\vec{E}_0{}^*\exp[i\int_0^x\kappa\,dx'-i\delta t], \tag{3}$$

where $v_1+3kv_e{}^2/\omega_1$, $v_2=c_s$, $\vec{k}=-\vec{k}_1(0)=\vec{k}_2(0)$, $\kappa=\sum_i\Delta k_i=\kappa' x$, and we have for convenience taken the density gradient and $\vec{k}_2$ in the x direction. Note that δ is not arbitrary, but rather a known function of k.

The resonance behavior of the component of the electric field parallel to the density gradient has been discussed by several authors.[4] Near resonance ($x=0$) one has

$$E_x/E_{\rm in}=\Phi(\tau)(2\pi k_0L)^{-1/2}L/[x+i\Delta], \tag{4}$$

where $L=n/n'$; $\Delta=(\lambda_D{}^2L)^{1/3}$, the resonance width, is due to finite electron temperature; $\tau=(k_0L)^{1/3}\times\sin\theta$; and $\Phi(\tau)$ is of order unity for a narrow range of angles of incidence.

Following Rosenbluth,[5] we examine Eq. (3) for the existence of temporally growing modes. Laplace transforming in time with p the Laplace transform variable, neglecting initial values, eliminating a_1, and letting

$$a_2=\psi^*\exp\left[i\int_0^x\frac{1}{2}\kappa\,dx+\frac{1}{2}\left(\frac{p^*+\nu_e}{v_1}-\frac{B^*+\nu_i}{v_2}\right)x\right], \tag{5}$$

where $B=(2i\omega_2)^{-1}(p^2+2i\omega_2p)$, we then find that

$$d^2\psi/dz^2+f(z)\psi=0, \tag{6}$$

where the function $f(z)$ is given by

$$f(z)=a^2/(z^2+\Delta^2)+\tfrac{1}{4}\kappa'^2(iz+d)^2 \tag{7}$$

and the WKB approximation is valid provided $\alpha\gg1$. Here

$$a=\frac{1}{2}\left(\frac{\Phi^2E_{\rm in}{}^2}{4\pi nT}\right)^{1/2}\frac{(2\pi k_0L)^{-1/2}L}{\lambda_D},\quad d=\frac{1}{\kappa'}\left(\frac{p+\nu_e}{v_1}+\frac{B+\nu_i}{v_2}+\frac{i\delta}{v_1}-\frac{1}{z-i\Delta}\right).$$

We have neglected corrections to $f(z)$ of order a^{-1}, ω_i/ω_e, and corrections proportional to ν_i.

Equation (6) has four complex turning points. The position of the turning points depends on the magnitude of the pump field and also on the relative magnitude of d, Δ. In Fig. 1 are shown the turning points and anti-Stokes lines for some representative values of these parameters. We will find that the turning points z_1 and z_2 are located on the real axis for the most rapidly growing modes.

Examining Eq. (6) for large $|z|$ we easily find that there are solutions which are well behaved at $z=\pm\infty$, namely,

$$\psi^{\rm I}(z)=f^{-1/4}(z)\exp[i\int_{z_1}^z f^{1/2}(z')\,dz'],\quad \psi^{\rm II}(z)=f^{1/4}(z)\exp[-i\int_{z_2}^z f^{1/2}(z')\,dz'], \tag{8}$$

where z_1 and z_2 are the turning points in the upper half-plane. These give solutions to Eq. (3) which behave for large $|z|$ as

$$a_1{}^{\rm I}(z)\sim z^{-1/2}\exp\{[-(p^*+\nu_e)/v_1]z\},\quad a_1{}^{\rm II}(z)\sim z^{-1/2}\exp\{[(B^*+\nu_i)/v_2]z\}. \tag{9}$$

Thus spatially localized but temporally growing modes exist provided an eigenvalue p can be found with Re$p>0$, Re$B>0$. The eigenvalue equation for p is

$$(n+\tfrac{1}{2})\pi=\int_{z_2}^{z_1}f^{1/2}(z)\,dz. \tag{10}$$

We look for an eigenvalue p comparable in magnitude to ω_2, in which case the $(z-i\Delta)^{-1}$ term in d can be neglected. From the definition of d we have $|d|\gg\Delta$ provided $k\Delta\gg1$, which we assume. In this case if the turning points are located such that $|z_t|\ll|d|$, a condition we must verify *a posteriori*, the function $f(z)$ can be approximated in the region between the turning points z_1, z_2

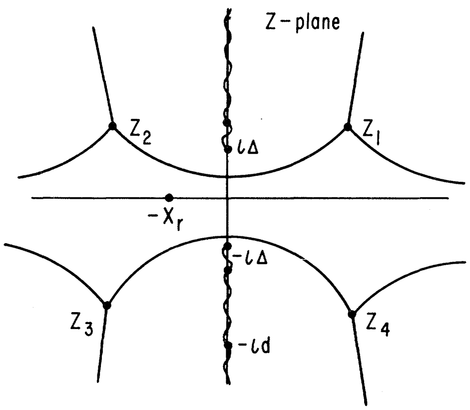

FIG. 1. Turning points and anti-Stokes lines for the differential equation $d^2\psi/dz^2 + f(z)\psi = 0$.

through

$$f(z) = a^2/(z^2+\Delta^2) - \tfrac{1}{4}\kappa'^2 d^2. \tag{11}$$

The other turning points, z_3 and z_4, are located far away in the lower half plane.

The integral in Eq. (10) can then be evaluated in terms of complete elliptic integrals and the eigenvalue p determined. By definition of the turning points we can write

$$f(z) = a^2/(z^2+\Delta^2) - a^2/(z_t^2+\Delta^2), \tag{12}$$

and, substituting into Eq. (10), we find $\mathrm{Im} z_t = 0$ and thus

$$d = 2a/\kappa'\Delta[1 + z_t^2(a)/\Delta^2]^{1/2}. \tag{13}$$

In Fig. 2 is shown the obtained relation between z_t/Δ and the pump strength a. The limiting range $z_t/\Delta \ll 1$ is readily found from Eqs. (10) and (12) to be

$$z_t/\Delta \simeq [(2n+1)/a]^{1/2}. \tag{14}$$

Note from Fig. 2 that only modes with $n > a/2$ have a spatial extent greater than Δ. Substituting the expression for d into Eq. (13), we find for the growth rate $\mathrm{Re}p \equiv \gamma$ and frequency shift $\mathrm{Im}p \equiv \Delta\omega$ the equations

$$\frac{\gamma}{\omega_2}\left[\omega_2\left(\frac{1}{v_1}+\frac{1}{v_2}\right)+\frac{\Delta\omega}{v_2}\right] = \frac{2a}{\Delta(1+z_t^2/\Delta^2)^{1/2}} - \frac{\nu_e}{k\lambda_D^2\omega_e},$$
$$\frac{(\Delta\omega)^2}{2\omega_2 v_2} + \Delta\omega\left(\frac{1}{v_1}+\frac{1}{v_2}\right) - \frac{\gamma^2}{2\omega_2 v_2} + \frac{\delta}{v_1} = 0, \tag{15}$$

where we have neglected the ion collision frequency.

In Fig. 3 are plotted the growth rates for two cases of interest for laser-pellet fusion. The pellet scale length is taken to be 10^{-1} cm and the electron temperature $T_e = 10$ keV. These growth

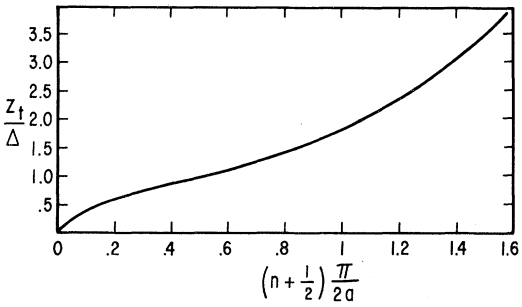

FIG. 2. z_t/Δ as a function of the pump strength a and mode number n. In the limit $z_t/\Delta \ll 1$, $z_t \approx \Delta[(2n+1)/a]^{1/2}$.

rates[2] are comparable to the maximum growth rates in a homogeneous plasma at the same values of pump amplitude and $k\lambda_D$, and the threshold for the existence of the modes ($a \gg 1$) is about equal to the threshold for spatial amplification of decay waves in the vicinity of the classical turning point where $\omega_e^2 = \omega_0^2\cos^2\theta$. The frequency shift $\Delta\omega$ is approximately equal to $-\frac{1}{2}\omega_2$ for small pump amplitude, increasing as the pump amplitude is increased and passing through zero when $\gamma \sim \omega_2$. For large pump amplitude $\gamma = \Delta\omega$ and $\gamma \sim a^{1/2}$. This gives $\gamma \sim E_0^{1/2}$ well above threshold, to be compared with $\gamma \sim E_0^{2/3}$ found[2] in the case of a homogeneous plasma. The growth rate increases for increasing k approximately as k^2.

In order to discuss energy absorption we must consider the effect of the turbulence which will produce an effective electron collision frequency ν_{eff}, modifying the width and maximum of the resonance through $\Delta = \nu_{\mathrm{eff}}L/\omega_p$ when $\nu_{\mathrm{eff}}/\omega_p > (\lambda_D/L)^{2/3}$. Assuming that a steady-state equilibrium maintains due to this increase in electron

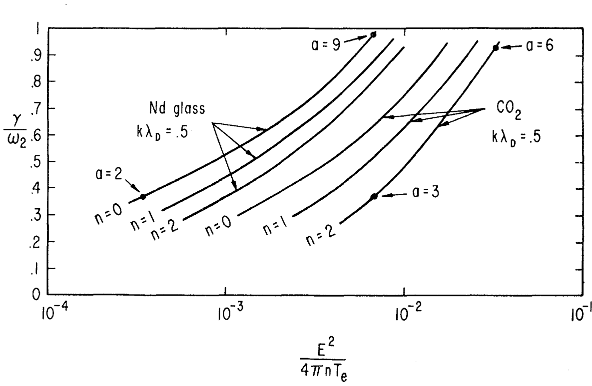

FIG. 3. Growth rate γ as a function of incident field strength. In the case of a Nd-glass laser we have taken $\lambda_D/L = 2.3\times10^{-5}$, and for a CO_2 laser $\lambda_D/L = 2.3\times10^{-4}$. Here and in Fig. 4 $\Phi(\tau)$ has been set equal to unity.

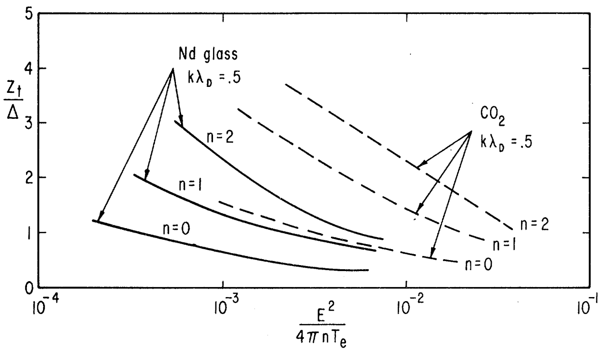

FIG. 4. z_t/Δ as a function of pump strength for the first few modes in the cases of CO_2 and Nd-glass laser-pellet irradiation. The scale parameters used are the same as those in Fig. 3.

collision frequency, we find from Eq. (15) that

$$\frac{\nu_{\text{eff}}}{\omega_p} = \left(\frac{\Phi^3 E_{\text{in}}^2}{4\pi n T}\right)^{1/4} (2\pi k_0 L)^{-1/4} (k_D)^{1/2} \tag{16}$$

which, for the examples considered, is larger than $(\lambda_D/L)^{2/3}$ for a significant range of angles of incidence. For the values of the parameters used, and using the functional form of $\Phi(\tau)$ from Ref. 4, we find $\nu_{\text{eff}}/\omega_p > (\lambda_D/L)^{2/3}$ and $\gamma/\omega_2 > 0.1$ for a range of angles of incidence $\delta\theta \approx 10°$ (Nd glass) and $\delta\theta \approx 15°$ (CO_2) when $E_2/4\pi n T = 10^{-2}$. For larger pump fields this window increases extremely slowly,

$$\delta\theta \sim \{\ln[2(L/\lambda_D)^{4/3}(E^2/4\pi n T)^{1/2}(2\pi k_0 L)^{-1/2} k\lambda_D]\}^{1/3},$$

because of the rapid decrease of $\Phi(\tau)$ for large τ. Following Kruer and Dawson,[6] the power absorbed is given by

$$P = \int \nu_{\text{eff}} [E^2(z)/4\pi]\, dz, \tag{17}$$

where the integration is to extend over the region of space in which the decay waves are unstable, i.e., between the turning points z_1 and z_2, and we include the kinetic energy of the plasma oscillations as well as the field energy. Using Eq. (4) we find the ratio of power absorbed to power incident on the plasma $cE_{\text{in}}^2/2\pi$ to be

$$A \approx (2\pi)^{-1} \arctan(z_t/\Delta) \tag{18}$$

which has a maximum value of $\frac{1}{2}$ when $z_t \gg \Delta$. See Fig. 4.

Preliminary experiments by Wong[7] using a microwave source have shown the existence of electron-plasma and ion-acoustic waves propagating parallel to the density gradient consistent with the results of the present work.

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