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Comparison of Nuclear and Coulomb Measurements of Nuclear Shapes*

D. L. Hendrie
Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720
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A calculation is described which contains hitherto neglected terms in the extraction of nuclear shapes from scattering data. These corrections are used in the comparison of deformations determined by measurements of distributions of nuclear potential with those of charge distributions, and serve to reduce the apparent discrepancies between those two types of measurements.

It has been well established that permanently deformed nuclei often have shapes that are more complicated than simple spheroidal deformations. These shapes were first accurately measured in the nuclear potential by scattering of α particles with energies well above the Coulomb barrier and for the rare-earth nuclei.[1] A systematic trend of hexadecapole deformation was discovered. Since then these basic results have been confirmed by a number of other experiments using other projectiles and energies,[2-8] have been extended to other regions of the periodic table,[9,10] and have been described by several theoretical treatments.[11,12] The experiments can be classified into two major categories, those that measure the shape of the nuclear potential[1-3,8,9] and those that measure the charge deformation.[4-8,10]

A simple and usual way of characterizing these deformations is to describe an appropriate nuclear radius in a multipole expansion

$$R = R_0(1 + \beta_2 Y_{20} + \beta_4 Y_{40} + \beta_6 Y_{60} + \cdots), \tag{1}$$

where the Y_{L0} are spherical harmonics and the β_L are the experimentally determined deformation parameters. The experiments all measure transition probabilities between states of the rotational band built on the ground state, since these probabilities are sensitively predicted by the nuclear shape in the rotational model. Complicated avenues of excitation are included by means of the coupled-channels calculations for nuclear excitation[13] and the Winter–de Boer code[14] for Coulomb excitation. Deviations from and additions to the simple rotational model can also be included, if found to be necessary.

A puzzling discrepancy has become apparent between the nuclear and Coulomb experimental results, in that the Coulomb work systematically finds larger values of hexadecapole deformations. This would, of course, be of basic importance if verified, since it implies a difference between the proton and neutron distributions at the nuclear surface. A long-standing difficulty in the comparison has been due to the different radii that characterize the two types of experiments. The Coulomb radius has been accurately measured by electron scattering to be about $1.1A^{1/3}$ fm for a suitably diffuse radial charge distribution, whereas the optical potential radius of Ref. 1, for example, was $1.44A^{1/3}$ fm, where A is the atomic mass of the target nucleus. Since the transition amplitudes depend sensitively on the radius, scaling of the measured β's with their corresponding radii must be done with care. Traditionally, this scaling has been accomplished using a suggestion of Blair[15] that the product $\beta_L R_0$ is a constant. This note will show, using a very simple model for the nuclear interaction, the origin of the simple scaling law, and also that significant higher-order effects occur which serve to reduce the discrepancy between the nuclear and Coulomb results for β_4.

A complete description of the interaction between a complex projectile and a deformed nucleus is not simple. Microscopically in lowest order, one would sum the realistic interactions between the nucleons in the projectile and the A target nucleons, which might be described in a Hartree-Fock calculation, for example. This has not yet been done.[16] Macroscopically, one would fold the projectile and target mass distributions with a finite-range interaction, including the possibility of an L dependence in the interaction.[17] Neither has this been done.[18] For the purpose of the present work, a much simpler model has been chosen. A spherical projectile is assumed to interact with a deformed nucleus only at their mutual sharply defined edges. However, from this picture we can extract geometric relationships that have immediate application but still would be common to any more realistic calculation.

From Fig. 1, let $R(\theta)$ describe the edge of a deformed target nucleus, and $r(\theta)$ describe the locus of the center of a projectile of radius Δ which just touches the nuclear surface. We define

$$r(\theta) = r_0[1 + \beta_2 Y_{20}(\theta) + \beta_4 Y_{40}(\theta) + \beta_6 Y_{60}(\theta) + \cdots] \quad (2)$$

and

$$R(\theta) = R_0[1 + \epsilon(\theta)], \quad (3)$$

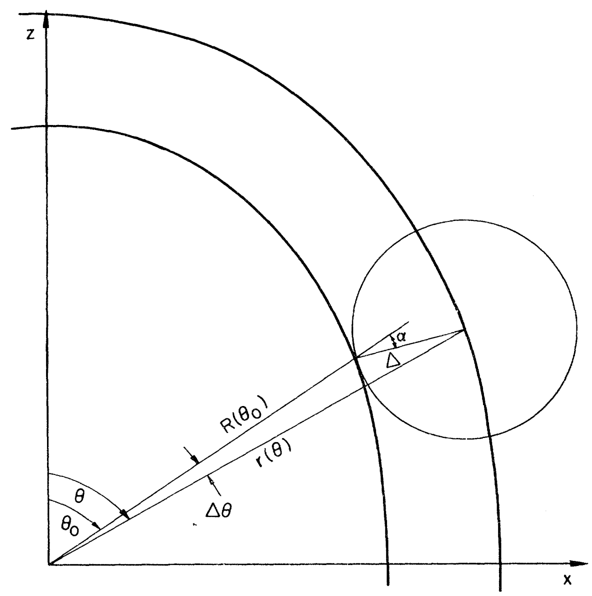

FIG. 1. Geometric quantities as described in the text. $R(\theta)$ defines the nuclear edge, and Δ is the radius of the projectile.

where

$$\epsilon(\theta) = \beta_{20}Y_{20}(\theta) + \beta_{40}Y_{40}(\theta) + \beta_{60}Y_{60}(\theta). \quad (4)$$

We wish to compare the values of r_0, β_2, β_4, β_6, etc. with the values of R_0, β_{20}, β_{40}, and β_{60}.

From the construction of Fig. 1, we have the angle α defined as the angular difference between the direction of $R(\theta_0)$ and the normal to the nuclear surface at θ_0. From the differential geometry we obtain

$$\tan\alpha = -R'(\theta_0)/R(\theta_0) = -\epsilon'(\theta_0)/[1 + \epsilon(\theta_0)]. \quad (5)$$

We expand $R(\theta) = R(\theta_0) + (dR/d\theta)|_{\theta_0}\Delta\theta + \cdots$, so that

$$\epsilon(\theta_0) = \epsilon(\theta) - \epsilon'(\theta)(\Delta\theta) + \cdots, \quad (6a)$$

$$\epsilon'(\theta_0) = \epsilon'(\theta) - \epsilon''(\theta)(\Delta\theta) + \cdots. \quad (6b)$$

From the trigonometric relationships, we obtain

$$r^2(\theta) = R^2(\theta_0) + \Delta^2 + 2\Delta R(\theta_0)\cos\alpha \quad (7)$$

and

$$\sin(\Delta\theta)/\Delta = \sin(\alpha)/r(\theta). \quad (8)$$

From Eqs. (5) and (6) we have, to lowest order in the small parameter ϵ',

$$\alpha \approx \tan\alpha \approx \sin\alpha = -\epsilon'(\theta), \quad \cos\alpha \approx 1 - \tfrac{1}{2}\epsilon'(\theta)^2, \quad (9)$$

and from Eqs. (8) and (9) we obtain, also to low-

TABLE I. $C_{ij}{}^L$.

$L \backslash ij$	22	24	26	44	46	66
0	0.239	0	0	0.796	0	1.671
2	0.270	2.418	0	1.393	6.680	3.131
4	−0.484	0.492	3.340	0.685	0.747	2.028
6	0	−1.908	0.983	−0.071	1.267	1.202

est order in ϵ and ϵ',

$$\Delta\theta \approx \sin(\Delta\theta) \approx [-\delta/(1+\delta)]\epsilon'(\theta), \tag{10}$$

where $\delta = \Delta/R_0$. Combining Eqs. (6), (7), (9), and (10), we obtain

$$r(\theta) = R_0\{1+\epsilon(\theta)+\delta+\tfrac{1}{2}[\delta/(1+\delta)]\epsilon'(\theta)^2\}, \tag{11}$$

which is correct to second order in ϵ and ϵ'. Finally, to obtain the values of r_0 and β_L, we multiply both sides by Y_{L0} and integrate over the sphere:

$$r_0 = R_0\left[1+\delta+\frac{1}{2}\frac{\delta}{1+\delta}\frac{1}{(4\pi)^{1/2}}\int Y_{00}\epsilon'(\theta)^2 d\Omega\right], \tag{12a}$$

$$\beta_L = \frac{R_0}{r_0}\left[\beta_{L0}+\frac{1}{2}\frac{\delta}{1+\delta}\int Y_{L0}\epsilon'(\theta)^2 d\Omega\right]. \tag{12b}$$

We define the constants $C_{ij}{}^L$ to yield the following:

$$r_0 = R_0\left[1+\delta+\frac{\delta}{1+\delta}\sum_{ij} C_{ij}{}^0\beta_{i0}\beta_{j0}\right], \tag{13a}$$

$$\beta_L = \frac{R_0}{r_0}\left[\beta_{L0}+\frac{\delta}{1+\delta}\sum_{ij} C_{ij}{}^L\beta_{i0}\beta_{j0}\right]. \tag{13b}$$

The leading term in Eq. (13b) gives immediately the scaling rule linear in R_0 as proposed by Blair. We note also that the origin of the linear scaling, rather than the $R_0{}^{2L}$ scaling that characterizes the electromagnetic moments, arises from the surface nature of the reaction. The radial corrections to the calculation of the deformations due to diffuse surface interactions are correctly handled by the reaction programs. To

TABLE II. List of deformation parameters.

		^{152}Sm	^{154}Sm	^{158}Gd	^{166}Er	^{174}Yb	^{176}Yb	^{178}Hf	^{182}W	^{238}U
Nuclear Excitation[a]										
$r_0 = 1.44\ A^{1/3}$ fm										
	β_{20}	0.205	0.225	0.235	0.230	0.230	0.230	0.205	0.190[d]	0.190[e]
	β_{40}	0.040	0.045	0.030	0	−0.040	−0.045	−0.060	−0.060	0.045
	β_{60}	−0.010	−0.015	−0.015	−0.015	0	−0.005	0	0	−0.015
1st Order										
$R_0 = 1.1\ A^{1/3}$ fm										
	β_2	0.268	0.295	0.308	0.301	0.301	0.301	0.268	0.249	0.249
	β_4	0.052	0.059	0.040	0	−0.052	−0.059	−0.079	−0.079	0.059
	β_6	−0.013	−0.020	−0.020	−0.020	0	−0.006	0	0	−0.020
2nd Order										
$R_0 = 1.1\ A^{1/3}$ fm										
	β_2	0.256	0.280	0.295	0.295	0.303	0.304	0.274	0.254	0.237
	β_4	0.061	0.071	0.053	0.015	−0.041	−0.046	−0.069	−0.070	0.067
	β_6	−0.006	−0.010	−0.013	−0.018	−0.007	−0.014	−0.009	−0.009	−0.012
Coulomb Excitation[b]										
$R_0 = 1.1\ A^{1/3}$ fm										
	β_2	0.286	0.315	0.330	0.344[g]	0.332[g]				0.261[f]
	β_4	0.068	0.066	0.030	−0.016	−0.030				0.106
Theory										
$R_0 = 1.1\ A^{1/3}$ fm[c]										
	β_4	0.076	0.083	0.063	0.024	−0.021	−0.032	−0.033	−0.047	0.071

[a]Ref. 1. [b]Ref. 6. [c]Ref. 11. [d]Ref. 3. [e]Ref. 9. [f]Ref. 10 [g]Ref. 7.

within the accuracy of the second terms included, the angular corrections to the deformations calculated here are independent of and in addition to the radial contributions arising from surface diffuseness.

A tabulation of the coefficients C_{ij}^L is given in Table I. We use these results to scale the results of Ref. 1 ($r_0 = 1.44A^{1/3}$ fm) to an appropriate Coulomb radius ($R_0 = 1.1A^{1/3}$ fm). This yields the reasonable value for the α-particle radius Δ to be 1.87 fm for ^{166}Er. Table II shows the original measurements, and the results after both first- and second-order scaling. Shown for comparison are some corresponding Coulomb-excitation results. The comparisons between the β_4's are improved to agree within experimental errors, with only slight changes for the β_2's. The β_6 values are greatly changed, but no comparisons are yet available. It is still probably premature to draw inferences from comparisons such as this now, however, since the spreads in published Coulomb-excitation values are far greater than the apparent discrepancies with the particle results. The comparison of the particle results with theoretical predictions of Ref. 11, however, is significantly improved. Checks on the size of the corrections from third- and higher-order terms show that they were not significant for β_2 and β_4. This may not be true for β_6, however, as is already indicated by the large corrections from the second-order terms. Because of the large projectile radii, forthcoming inelastic scattering experiments using heavy ions should be most sensitive to the scaling presented here.

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