# Upper Bound for the Sum of $\pi^+p$ and $\pi^-p$ Total Cross Sections above 60 GeV

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(Received 5 December 1973)

Using available experimental data, we give an upper bound for the sum of $\sigma_{\rm tot}(\pi^+p)$ and $\sigma_{\rm tot}(\pi^-p)$ above 60 GeV. It rules out an appreciable increase of this sum beyond 60 GeV.

High-energy experimental results in the last decade indicate that the Regge model is a useful but not necessarily an economical tool for the parametrization of high-energy experimental data. Because of its flexibility it can usually serve the purpose of correlating data and of giving a rough guide for planning future experiments. No fundamental importance can be attached if there is a disagreement between experimental results and the Regge-pole "prediction." This situation is understandable when dealing with complicated physical phenomena, where many variables are simultaneously involved (e.g., energy and momentum transfer). But for such a simple thing as the forward dispersion relation, it is very much desirable to adopt an approach where the flexibility is eliminated. This is the purpose of this note.

Using the averaged forward dispersion relation for the symmetric $\pi N$ amplitude and using the positivity of the total cross sections, the following upper bound is obtained:

$$\frac{1}{2\pi^2}\int_{M_1}^{M} dE'\,\sigma_s(E')\left[\frac{E'}{2(E_2-E_1)}\ln\frac{(E'+E_2)(E'-E_1)}{(E'-E_2)(E'+E_1)}-1\right]\leqslant 3.92\pm0.2 \text{ mb GeV}, \tag{1}$$

where $2\sigma_s(E)=\sigma_{\rm tot}(\pi^+p)+\sigma_{\rm tot}(\pi^-p)$, $E_1=8$ GeV, and $E_2=20$ GeV; $M_1<M$ and $[M_1,M]$ is any energy interval above 60 GeV. In practice this inequality is most useful for $M_1=N=60$ GeV, and $M$ is the highest energy attainable by available accelerators. Throughout this note, we shall set $M_1=N$. To have a feel for the usefulness of inequality (1), let us suppose that for $60\leqslant E\leqslant M$, we can parametrize $2\sigma_s=a+b\ln^2E/E_0$.

Let us first set $b=0$; then inequality (1) implies that $a$ must be smaller than $50.3\pm3$, $46.5\pm3$, and $44.0\pm3$ mb for $M=500$, 1000, and 5000 GeV, respectively. Alternatively, by setting $a=47.4$ mb (the Serpukhov data at 60 GeV) and $b=0$, then inequality (1) implies that $0.70\leqslant0.74\pm0.04$, $0.747\leqslant0.74\pm0.04$, and $0.785\leqslant0.74\pm0.04$ for $M=500$, 1000, and 5000 GeV, respectively.

If we now set $b\neq0$, $a=47.4$ mb, and $E_0=60$ GeV, then the energy-dependent coefficient $b$ is found to be smaller than $3.4\pm3.4$, $-0.38\pm2.24$, and $-1.47\pm1.3$ mb for $M=500$, 1000, and 5000 GeV, respectively.

From these results, it is clear that inequality (1) rules out any appreciable increase of $\sigma_s$ between 60 and 500 GeV. It tends to favor a constant behavior or a slight decrease in the total cross sections $\sigma_s$. Inequality (1) is insensitive to the values of $\sigma_s$ beyond 500 GeV [see Eq. (9) below]. It should be noticed that our result is also useful to test asymptotic *models* for the total cross sections; in this case we can set $M=\infty$.

Consider the $\pi^\pm p$ forward elastic amplitudes $f^\pm(E)$, where $E$ is the laboratory energy of the incident pions with momentum $q$, $E=(q^2+\mu^2)^{1/2}$, with $\mu$ the pion mass. Let $2f_s(E)=f^+(E)+f^-(E)$ and $2\sigma_s=\sigma^+(E)+\sigma^-(E)$. As a consequence of analyticity, crossing symmetry, and the Froissart-Martin upper bound[1] which are derived from the axiomatic field theory, the forward dispersion relation for $f_s(E)$ is valid and needs at most one subtraction (in the $E^2$ variable):

$$\begin{aligned}\mathrm{Re}f_s(E)-\mathrm{Re}f_s(\mu)-\frac{f^2}{M}\frac{E^2-\mu^2}{E^2-(\mu^2/2M)^2}\left(1-\frac{\mu^2}{4M^2}\right)^{-1}-\frac{E^2-\mu^2}{2\pi^2}\,\mathrm{P}\int_\mu^N\frac{dE'}{E'}\frac{(E'^2-\mu^2)^{1/2}}{(E'^2-E^2)}\sigma_s(E')\\ =\frac{E^2-\mu^2}{2\pi^2}\,\mathrm{P}\int_N^\infty\frac{dE'}{E'}\frac{(E'^2-\mu^2)^{1/2}}{E'^2-E^2}\sigma_s(E').\end{aligned} \tag{2}$$

Let us denote by $J_s(E)$ the integral on the left-hand side (lhs) and $I_s(E)$ the integral on the right-hand side (rhs) of (2). $N$ is chosen to be the highest energy where experimental data on $\sigma_s$ are available and satisfies the condition $E<N$. Here we set $N=60$ GeV. The lhs of (2) is expressed in terms of mea-

surable quantities. Our procedure consists of studying the integral $I_s(E)$ as a function of $E$ without making any assumption on $\sigma_s$ beyond the energy $N$. Since $\sigma_s$ is positive and $E<N$, is is clear that

$$I_s(E) \geqslant \frac{E^2-\mu^2}{2\pi^2}\int_{M_1}^{M}\frac{dE'}{E'^2-E^2}\,\sigma_s(E') \tag{3a}$$

$$\geqslant \frac{E^2-\mu^2}{2\pi^2}\int_{M_1}^{M}\frac{dE'}{E'^2}\,\sigma_s(E'). \tag{3b}$$

In practice, it is not convenient to use Eq. (2), because it is necessary to assume that $\sigma_s$ has a smooth behavior in order to carry out the principal-part integration. Although this is a currently accepted procedure, we would like to avoid making such an assumption.[2] For this purpose, we average Eq. (2) with an appropriate weight factor $h(E)$ in the energy interval $E_1$ to $E_2$ ($E_1<E_2<N$). Later we shall choose $E_1=8$ GeV and $E_2=20$ GeV. For a reason which will become clear later, we set $h(E)=1$.

Instead of Eq. (2), we now have

$$\mathrm{Re}g_s(E_1,E_2)=(E_2-E_1)^{-1}\int_{E_1}^{E_2}\mathrm{Re}f_s(E)\,dE \tag{4a}$$

$$=\mathrm{Re}f_s(\mu)+\frac{f^2}{M}+\frac{1}{2\pi^2}\int_{\mu}^{\infty}dE'\,\frac{E'}{q'}\,\sigma_s(E')K(E';E_1,E_2), \tag{4b}$$

with

$$K(E';E_1,E_2)=\frac{E'}{2(E_2-E_1)}\ln\left(\frac{E'+E_2}{E'+E_1}\right)\left|\frac{E'-E_1}{E'-E_2}\right|-1, \tag{4c}$$

where we have interchanged the order of integration, which is legitimate since (2) converges,[3] and have simplified the pole term by neglecting $\mu$ compared with $E_1$ and $E_2$. Unlike the factor $1/(E'-E)$ which appears in the principal-part integration, the function $K(E';E_1,E_2)$ is much easier to handle and the value of the integral in (4b) is therefore not sensitive to the experimental errors of $\sigma_s$. We now split the integral on the rhs of (4a) into two parts and define

$$\langle J_s(E_1,E_2)\rangle=\tfrac{1}{2}\pi^{-2}\int_{\mu}^{N}dE'\,(E'/q')\sigma_s(E')K(E';E_1,E_2), \tag{5a}$$

$$\langle I_s(E_1,E_2)\rangle=\tfrac{1}{2}\pi^{-2}\int_{N}^{\infty}dE'(E'/q')\sigma_s(E')K(E';E_1,E_2). \tag{5b}$$

From (4b) we thus obtain

$$\langle I_s(E_1,E_2)\rangle=\mathrm{Re}g_s(E_1,E_2)-\mathrm{Re}f_s(\mu)-f^2/M-\langle J_s(E_1,E_2)\rangle. \tag{5c}$$

The rhs of (5c) is therefore a direct measure of the integral (5b) which involves all the total cross sections from $N$ to infinite energy. Its value is given below. Since $\sigma_s(E')>0$ and $K(E';E_1,E_2)>0$ in the range of integration in (5b), we get the following inequality:

$$\langle I_s(E_1,E_2)\rangle>\tfrac{1}{2}\pi^{-2}\int_{M_1}^{M}dE'\sigma_s(E')K(E';E_1,E_2), \tag{6}$$

where $M_1, M>N$. In particular, to get a strict inequality we choose $M_1=N$. Equation (6) becomes

$$\tfrac{1}{2}\pi^{-2}\int_{N}^{M}dE'\,\sigma_s(E')K(E';E_1,E_2)<\mathrm{Re}g_s(E_1,E_2)-\mathrm{Re}f_s(\mu)-f^2/M-\langle J_s(E_1,E_2)\rangle. \tag{7}$$

Using experimental data for the real part of $f^{\pm}(E)$ in the 8–20-GeV energy region[4] and for $\sigma_s$ from the threshold to 60 GeV,[5] the $\pi$-$N$ coupling constant $f^2=0.078$, and $\mathrm{Re}f_s(\mu)=-0.002\pm0.004$,[6] the rhs of (7) or (5c) is calculated to be $3.92\pm0.2$ mb GeV as given in (1). This value is extremely insensitive to the values of $\mathrm{Re}f_s(\mu)$ and $f^2$. The experimental error quoted on the right-hand side of (1) is based on the assumption that the errors in $\mathrm{Re}f_s(E)$ are purely statistical. This may be optimistic.

A more transparent but somewhat less strict bound than (1) can be obtained from (7) by noting that in the range of integration the following inequality is valid:

$$K(E';E_1,E_2)>\tfrac{1}{3}(E_2^{\,2}+E_2E_1+E_1^{\,2})-(E')^{-2}. \tag{8}$$

Hence, we have the following bound for $\sigma_s$:

$$\int_{N}^{M}2\sigma_s(E')\,dE'/E'^2<0.74\pm0.04\ \text{mb GeV}^{-1}. \tag{9}$$

If the experimental values for $\mathrm{Re} f_s(E)$ were available for energy larger than 20 GeV, inequality (9) would lead to a poor bound for $\sigma_s$. In this case, we should use inequality (7).

We now comment on the choice of the weight function $h(E)$. Unlike previous considerations,[2,7] we want to amplify the high-energy contribution of (2) to improve the accuracy of the rhs of (1). This is so because the relative errors of the lhs of (2) decrease with increasing $E$. Hence the choice $h(E)=1$ was made.

To test the usefulness of our procedure, let us pretend that no information on $\sigma_s$ above 30 GeV is known. The rhs of inequality (1), now with $N=30$ GeV, gives $9.15\pm0.2$ mb GeV, which is larger by approximately a factor of 2 compared with the rhs of (1). Because the Serpukhov data obey this new inequality, we can conclude they are a test of the dispersion relation.

The implication from this discussion is clear: One cannot test dispersion relations by measuring the real part of the forward amplitude in the same energy range as that of the total cross section. Previous claims of tests of dispersion relations, using experimental data for the real part from 8 to 20 GeV and for the imaginary part below 30 GeV, are, in fact, tests of high-energy assumptions for the total cross section beyond 30 GeV.

We thank W. S. Lam for useful correspondence.

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