with the requirement of conservation of energy, this constitutes a selection rule so restrictive that no currents would be observable in our experimental range. In tunneling from axially symmetric vortex states into the plane-wave states of a normal metal, the selection rules, according to our preliminary calculations, are much less restrictive. Again there are difficulties in applying such a rule to a vortex in dirty material in which μ is not well defined, but we expect that an equivalent rule will apply, limiting the amount of tunneling, at least within a mean free path of the vortex center.

In conclusion, we believe that the vortex cores in our films are well aligned across our tunneling barriers, that the H^2-dependent tunnel currents in our S_1-I-S_2 junctions arise from those regions outside the smaller vortex cores whose properties are not independent of the vortex packing density, and, finally, that although the tunneling from a vortex core to a normal metal may show the usual probability, the tunneling between two vortex cores is reduced to negligible proportions by the requirement that angular momentum be conserved.

[1]M. Tinkham, Phys. Rev. 129, 2413 (1963).
[2]C. Caroli, J. Matricon, and P. G. de Gennes, Phys. Lett. 9, 307 (1964).
[3]R. J. Watts-Tobin and G. M. Waterworth, in Proceedings of the Thirteenth International Conference on Low Temperature Physics, Boulder, Colorado, 1972 (to be published).
[4]G. B. Donaldson, D. J. Brassington, and W. T. Band, to be published.
[5]J. Pearl, Ph. D. thesis, Polytechnic Institute of Brooklyn, 1965 (unpublished).
[6]C. J. Adkins, Phil. Mag. 8, 1051 (1963).
[7]G. B. Donaldson and D. J. Brassington, in *Proceedings of the Twelfth International Conference on Low Temperature Physics, Kyoto, 1970,* edited by E. Kanda (Keigaku Publishing Co., Tokyo, 1971), p. 429.
[8]J. E. Dowman, M. L. A. MacVicar, and J. R. Waldram, Phys. Rev. 186, 452 (1969).

Spin-Spin Correlations in an Ising Model for Which Scaling is Exact*

George A. Baker, Jr., and Geoffrey R. Golner
Applied Mathematics Department, Brookhaven National Laboratory, Upton, New York 11973
(Received 16 February 1973)

In their recent work using the renormalization-group approach, Fisher, Ma, and Nickel derived results for η when there is a long-range interaction, $r^{-d-\sigma}$, which were seemingly at variance with the numerical results of Nagle and Bonner. The problem is the presence of nonthermodynamic terms in the spin-spin correlation function which nevertheless contribute to the thermodynamics. We also find that the scaling relation between δ and η fails to hold for $d > 2\sigma$.

Recently Fisher, Ma, and Nickel[1] have computed theoretically the value of the critical index η for a system with long-range spin-spin interactions in d dimensions of the form $1/r^{d+\sigma}$. They employed the renormalization-group approach introduced by Wilson[2,3] and Fisher,[4] which has been rather successful in this general area. Consequently it is somewhat disturbing that these results ($d=1$, $\sigma \lesssim 0.3$) do not agree with the careful numerical estimates of Nagle and Bonner.[5] Baker[6] showed that there is a model, with long-range forces (but not translationally invariant), in which the approximate recursion relations, derived by Wilson[2] using the renormalization-group approach, are exact. Since this model is exactly solvable using renormalization-group methods, we have investigated its solution in order to attempt to resolve the discrepancy. We found that it is necessary for $\sigma/d < \frac{1}{2}$, when considering spin-spin correlations in this model, to distinguish between long long-range order and short long-range order. That is to say, we may have in the thermodynamic limit, when the system size tends to infinity, a different behavior when the length considered is long compared to the lattice spacing but short compared to the system size than we get when it is comparable to the system size. The quantity studied numerically by Nagle and Bonner,[5] called $\tilde{\eta}$ by them, is a characteristic of long long-range order, while the usual definition used by Fisher, Ma, and Nickel[1] is a characteristic of short long-range order. Our calculations reproduce both results in a highly satisfactory manner. They yield

$$\eta = 2-\sigma, \quad \tilde{\eta} = \min(2-\sigma, 2-\tfrac{1}{2}d) \tag{1}$$

for $0<\sigma<d$. There is no transition in the model for $\sigma>d$, and the total interaction strength per spin is infinite for $\sigma<0$ (see also Stell[7]). The result for η agrees with Suzuki's[8] inequality $\eta \geqslant 2-\sigma$, $0<\sigma<2$. We further find for the critical index δ ($H \propto M^{\delta}$ on the critical isotherm)

$$\delta = \max\left(3, \frac{d+\sigma}{d-\sigma}\right) = \max\left(3, \frac{d+2-\eta}{d-2+\eta}\right) = \frac{d+2-\tilde{\eta}}{d-2+\tilde{\eta}}. \tag{2}$$

The second two equalities follow from the first by Eq. (1). It is to be observed from Eqs. (1) and (2) that the Bragg-Williams approximation or "classical" results are obtained for $0<\sigma<\frac{1}{2}d$.

We have further investigated the question of symmetry of the susceptibility divergence exponents γ ($T>T_c$) and γ' ($T<T_c$), as well as the behavior of the spontaneous magnetization $[M \propto (T_c-T)^{\beta}]$ near the critical point. To check the validity of the Wilson[2] linearized recursion relation to evaluate γ we have used the thermodynamically prescribed procedure of first evaluating numerically the susceptibility χ at fixed temperature and then taking the limit as $T\to T_c$. The same type of procedure was used to study β. We found that the Wilson linearized recursion relation gives the thermodynamically correct value of γ for this model for all d. We further found

$$\gamma' = \gamma, \quad \beta = \gamma/(\delta-1). \tag{3}$$

The Ising model we study[6] is described by the Hamiltonian

$$H = J\sum_{\mu=0}^{Ld-1} 2^{-\mu\sigma/d} \sum_{m=1}^{2^{Ld-1-\mu}} s_{m,\mu}^{2} + mH2^{dL/2}\hat{s}_{1,Ld-1} - \tfrac{1}{2}J\,\frac{1-2^{-L(d+\sigma)}}{1-2^{-1-\sigma/d}} \sum_{j} \nu_j^{2}, \tag{4}$$

where

$$s_{m,\mu+1} = (\hat{s}_{2m-1,\mu} - \hat{s}_{2m,\mu})/\sqrt{2}, \quad \hat{s}_{m,\mu+1} = (\hat{s}_{2m-1,\mu} + \hat{s}_{2m,\mu})/\sqrt{2} \quad (m=1,\ldots,2^{Ld-2-\mu}),$$
$$\hat{s}_{j,-1} = \nu_j \quad (j=1,\ldots,2^{Ld}) \tag{5}$$

for $\mu=-1,0,1,\ldots,Ld-2$. The numerical studies have been done with the spins distributed as $\exp(+a\nu_j^{2}-1\nu_j^{4})$ to simulate spin-$\frac{1}{2}$ Ising-model spins, but other reasonable continuous distributions should yield similar results. This Hamiltonian can be interpreted[6] as representing a ferromagnetic Ising model on a hypercubical lattice of d space dimensions. However it should be borne in mind that the intrinsic structure is the same in all dimensions and the structure is the same as that of Dyson's hierarchial model.[9] As is evident from Eq. (7) below, the model depends only on (σ/d) and not on σ and d separately; therefore this model does not show the short-ranged behavior shown by the spherical model and conjectured by Nagle and Bonner[5] for $\sigma>2$ and large d. The effective spin-spin interaction decays in a stair-step fashion and behaves roughly like $1/r^{d+\sigma}$.

The partition function[6] is given by

$$Z = \prod_{\mu=0}^{Ld-1} [2^{0.5\sigma/d} I_\mu(0)]^{2^{Ld-1-\mu}} \int_{-\infty}^{+\infty} \exp\left[\beta mH2^{Ld/2}\hat{s}_{1,Ld-1} - \tfrac{1}{2}Q_{Ld}\frac{2^{1/2}\hat{s}_{1,Ld-1}}{2^{\sigma L/2}}\right] \frac{d\hat{s}_{1,Ld-1}}{2^{\sigma L/2}}, \tag{6}$$

where the recursion relations

$$I_\mu(x) = \int_{-\infty}^{+\infty} \exp[-Ky^2 - \tfrac{1}{2}Q_\mu(x+y) - \tfrac{1}{2}Q_\mu(x-y)]\,dy, \quad Q_{\mu+1}(x) = -2\ln[I_\mu(2^{-(1-\sigma/d)/2}x)/I_\mu(0)], \tag{7}$$

give the exact solution where $K=J/kT$ and $Q_0(X)$ is determined by the assumed distribution of the ν_j.

In this model every site is equivalent even though there is no translational invariance. Thus we can without loss of generality consider the $\langle \nu_1\nu_j\rangle$ spin-spin correlation function. By the use of Eq. (5) and the spin symmetry of (4), as reflected by the fact that the integral in (7) is even in y, we can compute directly that ($j>1$)

$$\langle \nu_1\nu_j\rangle = -2^{-l-1}\langle s_{1,l}^{2}\rangle + \sum_{k=l+1}^{Ld-1} 2^{-k-1}\langle s_{1,k}^{2}\rangle + 2^{-Ld}\langle \hat{s}_{1,Ld-1}^{2}\rangle, \tag{8}$$

where we define $l(j)$, $j>1$, by $2^{l}<j\leqslant 2^{l+1}$. It is to be noted from (8) and the corresponding result for

$j=1$ that the susceptibility per spin given by the sum rule

$$\chi/(m^2\beta N)=\sum_{j=1}^{2^{Ld}}\langle\nu_1\nu_j\rangle=\langle\hat{s}_{1,Ld-1}{}^2\rangle \tag{9}$$

is in agreement with the results of Ref. 6.

It follows from the definitions of the quantities involved that the calculation of $\langle s_{1,l}{}^2\rangle$ is obtained by changing the lth recursion relation [Eq. (7)] by inserting $2^{\sigma l/d}y^2$ in the integrand and proceeding as before. The expected value $\langle s_{1,l}{}^2\rangle=Z_l/Z$, where Z_l is the so-modified partition function. At the critical point it has been found[2,6] that the Q_μ tend to a limit as μ tends to infinity. Thus, in the short long-range limit we expect $\langle y^2\rangle_l$ to be independent of l. Hence

$$\langle\nu_1\nu_j\rangle\approx\langle y^2\rangle\left[\frac{2^{\sigma/d}-1}{2-2^{\sigma/d}}\right]2^{-(d-\sigma)l/d}+2^{-Ld}\langle\hat{s}_{1,Ld-1}{}^2\rangle. \tag{10}$$

To evaluate the last term we must analyze the behavior of the Q's in the range where they are bounded by some large but finite constant. From the recursion relations (7) it follows that $Q_\mu(x)$ is even in x and, for μ chosen large enough, has any required number of derivatives at the origin. For $\sigma/d>\frac{1}{2}$ it follows without difficulty that the Q's tend to a finite limiting function which behaves like $x^{2d/(d-\sigma)}$ for large argument.[6] For $\sigma/d<\frac{1}{2}$, if we recall that the critical point is characterized by the vanishing of the coefficient of $2^{\mu\sigma/d}x^2$ in Q_μ and consider $Q_\mu(x)$ for the expanding region $|x|\leq 2^{\mu(0.3-0.5\sigma/d)}$, then we find $Q_\mu(x)=A2^{\mu(2\sigma/d-1)}x^4$, where A is a nonzero constant, plus terms which vanish as μ goes to infinity.

Thus, we may evaluate Eq. (10) by use of Eq. (6) as

$$\begin{aligned}\langle\nu_1\nu_j\rangle&\approx\langle y^2\rangle\left[\frac{2^{\sigma/d}-1}{2-2^{\sigma/d}}\right]2^{-(d-\sigma)l/d}+\langle y^2\rangle_{Ld-1}2^{-L(d-\sigma)},\quad \sigma>\tfrac{1}{2}d,\\ &\approx\langle y^2\rangle\left[\frac{2^{\sigma/d}-1}{2-2^{\sigma/d}}\right]2^{-(d-\sigma)l/d}+\langle y^2\rangle_{Ld-1}2^{-Ld/2},\quad \sigma<\tfrac{1}{2}d.\end{aligned} \tag{11}$$

We note that even when $\sigma<\frac{1}{2}d$ and $Q_\mu\to 0$ for finite values of the argument, the term $\exp(-Ky^2)$ in the integrand of Eq. (7) maintains a finite value of $\langle y^2\rangle$. In the limit $d\to\infty$, the coefficient of $\langle y^2\rangle$ vanishes and a spin-spin correlation proportional to $1/\sqrt{N}$, where N is the number of spins, results. This result is exactly the same as the classical Bragg-Williams approximation.

We are now in a position to determine the index η. By definition, at the critical point,

$$\langle\nu_{\vec{0}}\nu_{\vec{r}}\rangle\propto 1/r^{d-2+\eta}, \tag{12}$$

and by the definition of distance[6] in this model $|\vec{r}|\propto 2^{l/d}$. Thus we conclude from (11) that $\eta=2-\sigma$ for $0<\sigma/d<1$ in accord with Eq. (1). The definition of Nagle and Bonner's[5] index $\tilde{\eta}$ is in terms of $\langle\nu_1\nu_{2^{Ld}}\rangle$. In this case the summation term in (8) is omitted and the analysis given above may be applied directly to the last term in Eq. (11). The conclusion is given in Eq. (1), which accords within the likely errors to the Nagle-Bonner data.

From Eqs. (5) and (6) and the definition of δ, it is clear that the index δ is determined by $Q_{Ld}(x)$ for values of x of the order of an arbitrarily small multiple of $2^{L(d-\sigma)/2}$. For $\sigma>\frac{1}{2}d$, the analysis of Ref. 6 for δ is valid, and we have confirmed $\delta=(d+\sigma)/(d-\sigma)$ by the direct numerical calculation of the asymptotic behavior of $Q_\mu(x)$ for large x. When $\sigma<\frac{1}{2}d$, the analysis we have given of $Q_\mu(x)$ after Eq. (10) suggests that $\delta=3$; however, the range of x treated is not large enough to be definitive. We have extended this range by direct numerical calculations and verified this result in the extended range. Together these conclusions give Eq. (2).

Equation (3) was found by evaluating the susceptibility and the magnetization in the thermodynamically prescribed manner, i.e. by taking first the limit $N\to\infty$, followed by $T\to T_c$. To facilitate this work we note that when the system size $2^{\mu/d}$ is large compared to the correlation length ξ, the spin-probability function $\exp[-Q_\mu(x)]$ must be of the structure $\exp[-2^\mu f(x)]$ when $f(x)$ is of the order of unity. Transforming our density function $Q_\mu(x)$ back to the scale of the original lattice size,

$$U_\mu(x)=2^{-\mu}Q_\mu(2^{\mu(d-\sigma)/2d}x), \tag{13}$$

we can then write the partition function of Eq. (6) as

$$Z(H,T)=Z_{Ld-1}(T)\int_{-\infty}^{\infty}dx\exp[-\beta mHNx-\tfrac{1}{2}NU_{Ld}(2^{1/2}x)], \tag{14}$$

where for $2^L \gg \xi$, $U_{Ld}(x)$ has minima[10] which depend only on T. For N large we can perform the final integral by using the saddle-point approximation. Expanding

$$U_{Ld}(2^{1/2}x)\approx U_{Ld}(2^{1/2}x_0)+U_{Ld}''(2^{1/2}x_0)(x-x_0)^2+\cdots, \tag{15}$$

we then have

$$\lim_{H\to 0}\chi(T)=\tfrac{1}{2}M^2\beta N/U_{Ld}''(2^{1/2}x_0), \tag{16}$$

where we have $x_0=0$ for $T>T_c$ and $x_0=M$, the reduced spontaneous magnetization per spin, for $T<T_c$.

Using the above formulas the spontaneous magnetization and the susceptibility were computed numerically to determine their temperature dependences above and below T_c. We find for $d>2\sigma$, $\gamma=\gamma'=1.000$, $\beta=0.500$ as expected. For $d=3$, $\eta=0.06$ as a comparison with the three-dimensional Ising model, we find $\gamma=\gamma'=1.256$, $\beta=0.3429$, with a numerical precision of ± 1 in the last place quoted for all the preceding numbers. Thus we find Eq. (3) to hold in these cases and all the other cases we have investigated numerically.

Using the techniques described above it is also possible to determine the spin-spin correlation function once the fluctuations become small. For $T>T_c$ we find,

$$\langle\nu_1\nu_j\rangle\approx\frac{\chi_0^{\,2}(1-2^{-\sigma/d})K}{2^{(d+\sigma)l/d}(1-2^{-(1+\sigma/d)})}, \tag{17}$$

where $\chi_0=\chi/(m^2\beta N)$ is the reduced susceptibility. Since $2^{l/d}\approx r$, we have a decay for r large, but still small compared to the system size, proportional to $r^{-(d+\sigma)}$ as expected,[7,8] for $0<\sigma<d$.

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[1]M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. 29, 917 (1972).

[2]K. G. Wilson, Phys. Rev. B 4, 3174, 3184 (1971).

[3]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).

[4]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. 28, 240 (1972).

[5]J. F. Nagle and J. C. Bonner, J. Phys. C: Proc. Phys. Soc., London 3, 352 (1970).

[6]G. A. Baker, Jr., Phys. Rev. B 5, 2622 (1972).

[7]G. Stell, Phys. Rev. B 1, 2265 (1970), and State University of New York at Stony Brook Report No. 231 (unpublished).

[8]M. Suzuki, University of Tokyo, Institute for Solid State Physics, Report No. ISSP-A-540 (unpublished).

[9]F. J. Dyson, Commun. Math. Phys. 12, 91 (1969).

[10]G. Golner, thesis, Cornell University (unpublished).

Experimental Study of Oscillatory Values of g^* of a Two-Dimensional Electron Gas

Amir A. Lakhani and P. J. Stiles
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 6 April 1973)

We have measured the surface electron density and the magnetic field dependence of the extremes of the oscillatory quasiparticle g factor in a two-dimensional electron gas.

An experimental approach with which one can measure the extreme values of the oscillations of the quasiparticle g value, g^*, of a two-dimensional electron gas (2DEG) is reported. We also report preliminary measurements which confirm this approach.

Electrons or holes can be confined to the surface of a semiconductor by the application of a sufficiently strong electric field normal to the surface. If the resulting potential well in the semiconductor is steep enough, then motion perpendicular to the surface will be quantized. At sufficiently low temperatures when the electron scattering time τ is large enough, such a system of electrons may behave as a 2DEG which has a density of states independent of energy. This two-dimensional nature of the surface electrons was shown by the experiments of Fowler *et al.*,[1] who studied Shubnikov–de Haas (SdH) oscillations of the electrons in inversion-layer conductivity in a Si metal-oxide-semiconductor (MOS) structure in high magnetic fields. They found a constant period of oscillation as a function of surface electron concentration n_s. The two-dimensional-