Parametric Instabilities in Bounded Plasmas

D. Pesme, G. Laval, and R. Pellat
Centre de Physique Théorique de l'Ecole Polytechnique, Paris, France
(Received 23 April 1973)

We have obtained conditions for excitation of parametric instabilities in bounded plasmas. Damping and inhomogeneity effects are taken into account. The results are found to be of special importance for backscatter Brillouin instability.

Plasma parametric instabilities occur when the pump amplitude exceeds a threshold which depends on collisional effects or Landau damping. It has been recently pointed out[1] that plasma inhomogeneities could also stabilize parametric instabilities by destroying the matching conditions between wave numbers and frequencies. These effects are thought to be important in the special case of backscatter instabilities in a laser-irradiated plasma. We want to demonstrate that another stabilizing influence is provided by the finite extent of the plasma region where the instability can occur. Firstly, we notice that inhomogeneities can stabilize the stimulated Brillouin scattering (SBS) only if a strong electron temperature gradient is present. Numerical computations[2] exhibit weak electron temperature gradients because of the large electronic thermal conductivity. Thus the effective threshold will be given in this case by the finite length of the medium. Secondly it must be emphasized that according to Rosenbluth's paper,[1] only spatial amplification can occur in an inhomogeneous plasma, so that the threshold depends on both the inhomogeneity and the size of the unstable region.

Therefore, in this paper we discuss the threshold for parametric instabilities in bounded plasmas. At first, by neglecting wave damping, stability conditions are obtained for both convective and absolute modes in a homogeneous bounded plasma. If the group velocity of any of the decay products is much larger than the other one, it is found that the critical length for instability is surprisingly short, but the normal-mode growth rates are reduced. In such a case, a weak damping can modify these results. When plasma inhomogeneity is taken into account, we have to compare the amplification length to the plasma length. Once more, a weak damping can increase drastically the amplification length though the net e-folding is not changed.

As in Ref. 1, we assume that three coherent waves $i=1, 2, 3$ are nonlinearly coupled and propagate along the x axis. In the small-amplitude approximation, the amplitudes of these waves can be written as

$$A_i = a_i(x,t)\exp[-i(\omega_i t - k_i x) + i\int_0^x \Delta k_i(x)\,dx] + \text{c.c.},$$

where the a_i are slowly varying functions of x and t, ω_i and k_i are the real frequencies and wave numbers which are linked by the linear dispersion equation at $x=0$. The frequencies are kept fixed, and the inhomogeneity is taken into account by $\Delta k_i(x)$ in the WKB approximation. We assume the resonance condition to be fulfilled at $x=0$, so that $\sum\omega_i = \sum k_i = \Delta k_i(0) = 0$. If wave 3 is the pump wave, the parametric instability is described by the equations

$$\partial a_1/\partial t + V_1 \partial a_1/\partial x + \Gamma_1 a_1 = \gamma_0 a_2^* \exp[i\int_0^x \mathcal{K}(x)\,dx], \tag{1a}$$

$$\partial a_2^*/\partial t + V_2 \partial a_2^*/\partial x + \Gamma_2 a_2^* = \gamma_0 a_1 \exp[-i\int_0^x \mathcal{K}(x)\,dx], \tag{1b}$$

where $\mathcal{K} = \sum_i \Delta k_i$; V_1, Γ_1 and V_2, Γ_2 are respectively the group velocities and the linear dampings of the two waves; γ_0 is the growth rate of the parametric instability in the homogeneous case and can be taken as real and positive.

At first let us neglect Γ_1, Γ_2, and $\mathcal{K}(x)$ and assume that γ_0 is not negligible in a region of finite extent l $(0<x<l)$. To look for normal modes, we set $a_1 = \alpha_1(x)e^{-i\Omega t}$ and $a_2^* = \alpha_2^*(x)e^{-i\Omega t}$. In order to fulfill boundary conditions at infinity for unstable normal modes with $\text{Im}\Omega > 0$, we must take $\alpha_i(x) = 0$ for $x<0$ if $V_i > 0$ and $\alpha_i(x)=0$ for $x>l$ if $V_i<0$. More generally, growing normal modes which would not fulfill these conditions at $x=0$ and $x=l$ would be physically meaningless since there would be an infinite energy input from the outside into the plasma for $t\to\infty$. Then if $V_1V_2>0$, we cannot find unstable normal modes. On the other hand, if $V_1V_2<0$, unstable modes can exist and the roots of the dispersion equation are given by the following relations:

$$\Omega = \gamma_0 |V_1 V_2|^{1/2} |V_1 - V_2|^{-1} (\xi + i\eta), \tag{2}$$

with $\xi = \pm 2 \sin\alpha l' \sinh\beta l'$ and $\eta = -2\epsilon \cos\alpha l' \cosh\beta l'$, $l' = l\gamma_0/|V_1 V_2|^{1/2}$, where α and β are solutions of the two equations

$$\sin\alpha l' \cosh\beta l' = \epsilon\alpha, \tag{3a}$$

$$\sinh\beta l' \cos\alpha l' = \epsilon\beta, \tag{3b}$$

with $\epsilon = \pm 1$ and $\alpha\beta \neq 0$. If $l' \leq \pi/2$, there is no unstable root. If $l' > \pi/2$, growing modes exist with $\mathrm{Re}(\Omega) = \xi = 0$. All unstable roots are such that $\xi = 0$, $\eta < 2$ and $\eta \simeq +2$ if $l' \gg \pi/2$. We conclude that absolute instability develops in the bounded unstable region if

$$\gamma_0 l/|V_1 V_2|^{1/2} > \pi/2, \tag{4}$$

but the growth rate remains always smaller than $2\gamma_0 |V_1 V_2|^{1/2}/|V_1 - V_2|$ which is precisely the growth rate of the absolute instability in the infinite homogeneous plasma.[3] From these results we deduce two interesting conclusions concerning the backscatter Brillouin instability (photon → photon + phonon). If the electron temperature gradient is negligible, the threshold will be given by

$$\gamma_0 > \pi(cC_s)^{1/2}/2l, \tag{5}$$

with

$$\gamma_0 = \alpha\omega_{pi}(\omega_3/\omega_s)^{1/2}, \tag{6}$$

where ω_{pi} is the ion plasma frequency, $\omega_s = k_2 C_s$ with C_s the ion sound speed, and $\alpha = eE_3/m_e\omega_3 c \ll 1$, E_3 being the pump-wave electric field amplitude. It is seen that the plasma is unstable even if γ_0 is smaller than the transit frequency of the fast electromagnetic wave through the plasma slab. Moreover, it is known[4] that the growth rate of the parametric instability in the infinite homogeneous case is given by (6) only when $\gamma_0 < k_2 C_s$ because $a_i^{-1}\partial a_i/\partial t$ has been assumed to be much smaller than ω_i in deriving Eq. (1). Nevertheless the values of normal-mode growth rates are much smaller than γ_0 so that the same approximation is still valid in the range $k_2 C_s < \gamma_0 < k_2 C_s (c/C_s)^{1/2}$. Thus, we can use the threshold (5) in that case.[5]

We recall that if $V_1 V_2 > 0$, we can have only convectively unstable perturbation. If we set $a_1 = \alpha_1 \exp(rx + i\Omega t)$ and $a_2{}^* = \alpha_2{}^* \exp(rx + i\Omega t)$ where Ω is now real, we readily find that r has the largest real part $\mathrm{Re}(r) = \gamma_0/|V_1 V_2|^{1/2}$ for $\Omega = 0$ which gives an instability criterion similar to the criterion (4) for normal modes ($V_1 V_2 < 0$). This case ($V_1 V_2 > 0$) is of importance for SBS if the plasma expansion velocity is larger than C_s.[5]

The values of the normal-mode growth rates are found to be much smaller than γ_0 if $|V_1/V_2| \gg 1$, so that these modes are expected to be easily stabilized by a weak damping. Therefore, we now keep the damping rates Γ_1 and Γ_2 in Eq. (1) but still neglect the mismatching $\mathcal{K}(x)$. We find then from the dispersion relation that damping does not affect the instability if

$$\gamma_0{}^2 \gg (|V_1 V_2|/4)(\Gamma_1/V_1 - \Gamma_2/V_2)^2, \tag{7a}$$

$$\gamma_0{}^2 \gg \Gamma_1 \Gamma_2. \tag{7b}$$

For backscatter instabilities we have normally $|\Gamma_1/V_1| \ll |\Gamma_2/V_2|$ where wave 1 is the electromagnetic backscattered wave and wave 2 the longitudinal one, so that conditions (7a) and (7b) become $\gamma_0 \gg \gamma_c \gg \gamma_T$, where $\gamma_T = (\Gamma_1\Gamma_2)^{1/2}$ is the usual threshold for the parametric instability growth rate in an infinite homogeneous medium[4] and $\gamma_c = (\Gamma_2/2)|V_1/V_2|^{1/2}$. In this regime the instability criterion reduces to condition (4). In the intermediate regime where $\gamma_T < \gamma_0 < \gamma_c$, we find only spatial amplification whatever the sign of $V_1 V_2$. The net e-folding $(\gamma_0{}^2/|V_1|\Gamma_2 - \Gamma_1/|V_1|)l$ gives then the excitation condition

$$(\gamma_0{}^2/|V_1|\Gamma_2 - \Gamma_1/|V_1|)l \gg 1. \tag{8}$$

This e-folding is smaller than the one in the undamped case: $\gamma_0 l/|V_1 V_2|^{1/2}$ since $\gamma_0 \ll \Gamma_2 |V_1/V_2|^{1/2}$. If $\gamma_0 < \gamma_T$, no instability can exist. Finally we see that the e-folding length depends strongly on the damping rate Γ_2 even if $\Gamma_2 \ll \gamma_0$, provided that $|V_2| \ll |V_1|$. For SBS, if we set $\Gamma_2 = \epsilon_s \omega_s$ ($\epsilon_s < 1$), the condition $\gamma_0 < \gamma_c$ becomes $\alpha < 2^{1/2}\epsilon_s \times (k_3 \lambda_D)$. This inequality can be fulfilled for moderate-power laser beams when the ratio of electron to ion temperatures is not very large. In this case, the condition for SBS excitation is

$$\gamma_0{}^2 l/c\Gamma_2 \gg 1 \tag{9}$$

when the electron temperature gradient is negligible.

We now consider the inhomogeneous plasma case and want to take the mismatching $\mathcal{K}(x)$ and bounded plasma effects into account. We specialize to the particular dependence $\mathcal{K}(x) = \mathcal{K}'(0)x$. If the damping is negligible and the plasma is unbounded, the e-folding has been found by Rosenbluth to be

$$\pi\gamma_0{}^2/|V_1 V_2 \mathcal{K}'|. \tag{10}$$

Then if $V_1 V_2 > 0$, the e-folding length is given by

$\gamma_0/|\mathcal{K}'|(V_1V_2)^{1/2}$ which has to be compared to the plasma length; if $V_1V_2<0$, it can be checked easily that the inhomogeneity does not affect the normal modes when $\gamma_0/|\mathcal{K}'||V_1V_2|^{1/2}\gg l$. Thus if this inequality is satisfied the instability condition reduces to

$$\gamma_0 l/|V_1V_2|^{1/2}\gg 1. \tag{11}$$

On the other hand if $\gamma_0/|\mathcal{K}'||V_1V_2|^{1/2}<l$, boundedness effects are negligible so that Rosenbluth's criterion

$$\pi\gamma_0^2/|V_1V_2\mathcal{K}'|\gg 1 \tag{12}$$

must be applied.

As shown previously damping is important when $|V_1|\gg|V_2|$ and $|\Gamma_1/V_1|\ll|\Gamma_2/V_2|$. Assuming in the following we are in this case, we now consider the effect of damping rates Γ_1 and Γ_2 on the excitation of parametric instabilities in the inhomogeneous bounded plasma. If $\gamma_0\gg\gamma_c\gg\gamma_T$, damping is again negligible, and we recover the previous results (11) or (12). In the intermediate regime $\gamma_T<\gamma_0<\gamma_c$, we must take damping rates Γ_1 and Γ_2 into account in Eq. (1). We Laplace transform in time Eqs. (1) with p the Laplace variable; and putting

$$a_1=\exp\{\tfrac{1}{4}i\mathcal{K}'(0)x^2-\tfrac{1}{2}[(p+\Gamma_2)/V_2+(p+\Gamma_1)/V_1]x\}F,$$

we find

$$d^2F/dx^2+(\tfrac{1}{4}\{\mathcal{K}'(0)x+i[(p+\Gamma_2)/V_2-(p+\Gamma_1)/V_1]\}^2+\tfrac{1}{2}i\mathcal{K}'(0)-\gamma_0^2/V_1V_2)F=0. \tag{13}$$

Only spatial amplification is possible in this case, and we can use the WKB approximation to solve Eq. (13). We put a small source at zero frequency at $x=0$ and the proper solution for a_1 is given by

$$a_1(x)=a_1(0)\exp\{-\Gamma_1x/V_1+(i\gamma_0^2/|V_1V_2\mathcal{K}'|)[\ln(x/x_c+i)-i\pi/2]\} \tag{14}$$

if $V_1>0$, $x>0$ and with $x_c=|\Gamma_2/V_2||\mathcal{K}'(0)|^{-1}$. Whatever the sign of V_1V_2, we obtain the same net e-folding (10) as in the undamped case provided that $\gamma_0>\gamma_T$. This result was already obtained by Perkins and Flick.[1] The amplification length x_c is much larger than in the undamped case as soon as $\gamma_0\ll\gamma_c$. If the plasma length l is smaller than x_c, the inhomogeneity plays no role and we recover the previous condition for the homogeneous bounded system (4) or (8).

We apply these results to the stimulated Brillouin scattering. The damping is negligible if $\gamma_0\gg\gamma_c$, i.e., $\alpha\gg 2^{1/2}\epsilon_s(k_3\lambda_D)$; and as soon as the inequality

$$\alpha L_T/k_3\lambda_D>l \tag{15}$$

is fulfilled, where $L_T=[(d/dx)\ln T_e]^{-1}$ is the inhomogeneity scale[6] and λ_D the Debye length, the bounded-plasma instability criterion (5) must be used, i.e., in terms of α,

$$\alpha(k_3l)>2^{-1/2}\pi(k_3\lambda_D). \tag{16}$$

In the intermediate regime $\alpha<2^{1/2}\epsilon_s(k_3\lambda_D)$, as soon as the inequality

$$\epsilon_s L_T>l \tag{17}$$

is fulfilled the bounded-plasma criterion (9) must be used, i.e.,

$$\alpha(k_3l)\gg(2^{1/2}k_3\lambda_D/\alpha)2^{3/2}(k_3\lambda_D). \tag{18}$$

In both cases, when the inequalities (15) or (17) are violated, the inhomogeneous unbounded-plasma excitation condition must be used, namely,

$$\tfrac{1}{4}\pi\alpha^2(k_3L_T)/(k_3\lambda_D)^2\gg 1. \tag{19}$$

Comparing inequalities (16), (18), and (19), we conclude that when the plasma boundedness dominates over the inhomogeneity, the instability conditions (16) and (18) are more severe than Rosenbluth's condition (19). It will be often the case if $L_T\gg l$.

Several of these results for SBS have been obtained independently by Forslund, Kindel, and Lindman.[7] The threshold for the undamped homogeneous bounded case was found by looking for a stationary solution of the full nonlinear set of equations for the three waves. Their critical length agrees with (16) within a numerical factor. Nevertheless, the temporal stability was not examined and the growth rate was not given. In the damped case these authors used the same method as ours. The importance of damping in the inhomogeneous case was not discussed.

Finally we emphasize that we have given the instability conditions for all possible cases taking damping, boundedness, and inhomogeneity into account. The results are of special importance when one of the two decay products is much slower than the other one whatever the sign of their group velocities.

The authors gratefully thank M. N. Rosenbluth

and R. Z. Sagdeev for enlightening discussions.

[1]F. W. Perkins and J. Flick, Phys. Fluids 14, 1012 (1971). This paper deals with the decay of an electromagnetic wave into Langmuir and ion-acoustic waves, and thresholds are given for an inhomogeneous unbounded plasma with damping. See M. N. Rosenbluth, Phys. Rev. Lett. 29, 565 (1972), where thresholds are given for any decay-type parametric instabilities in inhomogeneous dissipationless unbounded plasmas.

[2]J. S. Clarke, H. N. Fisher, and R. J. Mason, Phys. Rev. Lett. 30, 89 (1973).

[3]A. Bers and R. J. Briggs, Quarterly Progress Report No. 71, Research Laboratory of Electronics, Massachusetts Institute of Technology, 1963 (unpublished), p. 122.

[4]K. Nishikawa, J. Phys. Soc. Jap. 24, 916 (1968).

[5]For SBS, the exact dispersion equation can be obtained without the weak-coupling approximation $a_i^{-1} \times da_i/dt < \omega_i$. It can be checked that when $V_1V_2 < 0$, the absolute instability growth rate keeps approximately the same value $2\gamma_0(C_s/c)^{1/2}$ provided that $\gamma_0 < \omega_s(c/C_s)^{1/2}$ and the approximate dispersion relation can be used. If $V_1V_2 > 0$, in the same range $\gamma_0 < \omega_s(c/C_s)^{1/2}$ we find again $\mathrm{Re}(r) = \gamma_0/(cC_s)^{1/2}$ which insures $a_i^{-1}da_i/dx < k_i$.

[6]A. A. Galeev *et al.*, Pis'ma Zh. Eksp. Teor. Fiz. 17, 48 (1973) [JETP Lett. 17, 35 (1973)]. In this paper, it is pointed out that the inhomogeneity of the blowoff velocity U can be more important than the electron temperature inhomogeneity. In this case L_T should be replaced by $L_U = [d/dx \ln U]^{-1}$.

[7]D. W. Forslund, J. M. Kindel, and E. L. Lindman, Phys. Rev. Lett. 30, 739 (1973).

Argon Shear Viscosity via a Lennard-Jones Potential with Equilibrium and Nonequilibrium Molecular Dynamics*

W. T. Ashurst†
Sandia Laboratories, Livermore, California 94550

and

W. G. Hoover
Department of Applied Science, University of California at Davis, Davis, California 95616, and Lawrence Livermore Laboratory, Livermore, California 94550

(Received 30 May 1973)

Nonequilibrium molecular dynamic simulation of liquid argon yields the strain-rate dependence of shear viscosity. Near the triple point the apparent viscosity *decreases* with increasing strain rate; the extrapolated zero-gradient viscosity is consistent with the equilibrium Green-Kubo viscosity calculated by Levesque, Verlet, and Kurkijarvi. At higher temperatures along the saturated vapor pressure line, our results are insensitive to the strain rate and agree well with experimental data for liquid argon.

We have developed a *non*equilibrium molecular-dynamic method to simulate directly dense-fluid transport.[1] The shear-viscosity coefficient is determined from a Couette flow where the bounding planar fluid walls have steady relative velocity. Systems of 108 and 216 Lennard-Jones particles have been simulated[2] for real time durations (for argon) of 10^{-10} sec. The average flow velocity has a linear profile, and when divided into the wall shear stress determines the Newtonian shear-viscosity coefficient $\eta \equiv -P_{xz}/u_{xz}$. In Fig. 1 our results are compared with experimental argon shear viscosity[3-5] along the saturated vapor-pressure line of argon. The overall excellent agreement indicates successful simulation of nonequilibrium Couette flow with few-particle systems.

More extensive calculations have been made in the triple-point region for comparison with a recent equilibrium molecular-dynamic calculation by Levesque, Verlet, and Kurkijarvi (LVK).[6] These equilibrium calculations use the Green-Kubo relations to relate the transport coefficients to time correlations of the equilibrium fluctuations. An 864-atom Lennard-Jones system was studied for 10^{-9} sec (for argon) with a shear-viscosity coefficient of $\eta\sigma^2(m\epsilon)^{-1/2} = 4.02 \pm 0.3$ and thermal-conductivity coefficient of $\lambda\sigma^2(m/\epsilon)^{1/2}/k = 14.8$ at $N\sigma^3/V = 0.8442$ and $kT/\epsilon = 0.772$ (for argon, $\sigma = 3.405$ Å and $\epsilon/k = 119.8$°K).

Our nonequilibrium results for shear viscosity at the triple-point region depend upon the velocity gradient u_{xz}. See Table I. Thus while the highest velocity-gradient result is below the experimental argon results, the lowest velocity-gradient result approaches the equilibrium molecular-dy-