frequency polarizability tensor, $\Delta\alpha$, scales with other measures of $\langle P_2 \rangle$. The proportionality between $\Delta\alpha$ and nematic order is direct with no need for local-field corrections. Similarly, the Raman data for $\langle P_2 \rangle$ are in excellent agreement with other data for $\langle P_2 \rangle$ without introduction of local-field effects. The empirical fact that local-field corrections are not necessary for either of these measurements can be rationalized as follows: Local-field corrections depend on dipole-dipole summations over neighboring molecules. Although the problem can be solved in a number of ways, the answers are ultimately dependent on near-neighbor correlations. Near-neighbor correlations appear to be relatively insensitive to temperature implying the same about the local-field corrections. In the present case the ratio a/b in the isotropic phase, which should also reflect local-field effects, was measured to be independent of temperature and we assume the same value applies to the nematic phase. The $\langle P_2 \rangle$ data support this assumption.

[17] $\Delta\alpha$ was obtained from our refractive index measurements and the "Vuks equation" (see Ref. 8).

Comment on "π Condensation in Nuclear Matter"*

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Recent remarks on the possibility of negative-pion condensation in superdense nuclear matter are shown to be based on an incorrect formalism. The possibilities of π^- and π^0 condensation are compared and some problems are enumerated which must be solved before it is known which, if either, of the phenomena can occur.

In an interesting recent Letter, mainly concerned with neutral-pion condensation in superdense nuclear matter, Migdal has made some remarks on the possibility of a charged-pion condensation.[1] However, his treatment is based on a formal approach which is untenable for the charged-meson problem.

At the very beginning of Ref. 1 it is stated that the critical condition for π^- condensation in a neutron gas is given by the solution to the equation

$$\vec{k}^2 + m_\pi^2 - \mu^2 + \Pi(\mu, \vec{k}) = 0. \quad (1)$$

Here $\mu = \mu_n - \mu_p$ is the difference between the neutron and proton chemical potentials; k is the wave vector of the π^- mode; $\Pi(\omega, \vec{k})$ is the proper polarization part for the π^- propagating in the neutron medium. Migdal further states that in the limit of a vanishing number of pions (and protons), just above the critical density, the parameter $\mu = \mu_n - \mu_p$ will be equal to the Fermi energy of the neutrons.

I shall demonstrate below that Eq. (1) can be used to derive the critical density when the correct value of μ is used. But the correct value of μ is so different from ϵ_n^F that Migdal's remarks based on this equation have no relevance to the π^- problem. The chemical potential difference $\mu_n - \mu_p$ must be derived dynamically from the formula for the condensation energy. Even in the case of infinitesimal pion density, μ will differ from ϵ_n^F by a finite and large amount. Furthermore, one can look directly at the energy of the system to see that the solution of (1), with $\epsilon = \epsilon_n^F$, is unrelated to any phase transition.

Since the π^-, π^+ condensation problem has been treated elsewhere in some detail[2-4] for realistic values of the parameters, I discuss here a limit $M_N \to \infty$, in which the problem is trivially solvable and which illustrates the important points of principle.

The π^- polarization operator in a free Fermi gas of neutrons, calculated to second order in the pion-nucleon coupling constant f, comes directly from the crossed graph with an intermediate proton and is given by

$$\Pi(\omega, k) = -2f^2k^2\rho/\omega m_\pi^2, \quad (2)$$

where ρ is the density of neutrons. If one looks at Eq. (1), takes $\mu = \epsilon_F \to 0$ (as $M_N \to \infty$), and uses the expression for Π given in Eq. (2), it will be found that there will be a solution for arbitrarily small values of f, k, or ρ, in the limit $M_N \to \infty$. That is, in the static limit a π^- condensation is predicted for arbitrarily small coupling, or density.

To show that this is a defect of the formalism and not a pathology of the static limit, consider

the energy of a pion-condensed electrically neutral system, with the charged-pion field given by the function

$$\varphi(\vec{x}) = (\rho/2\omega_k)^{1/2}[(X+Y)^{1/2} + Y^{1/2}]e^{ikz} \quad (3)$$

and the canonical momentum given by[5]

$$\Pi(x) = i(\rho\omega_k/2)^{1/2}[(X+Y)^{1/2} - Y^{1/2}]e^{ikz}, \quad (4)$$

where $\omega_k = (\vec{k}^2 + m_\pi^2)^{1/2}$. These fields represent a mixture of a condensed mode of π^- with wave number $k\hat{z}$ and of π^+ with wave number $-k\hat{z}$. The quantity $(X+Y)\rho$ is defined as the density of π^- particles and $Y\rho$ as the density of π^+ particles, where ρ is now the total nucleon density. Thus if the system is to be electrically neutral, the proton density will be $X\rho$.

Let us now write the formula for the energy due to the presence of pions which comes directly from the Hamiltonian in our static model:

$$\langle H_\pi\rangle = \int d^3x\,[\Pi^*(\vec{x})\Pi(\vec{x}) + \nabla\varphi^*\cdot\nabla\varphi + m_\pi^2\varphi^*\varphi] + i2^{1/2}fkm_\pi^{-1}\int d^3x\,[\langle n^\dagger(\vec{x})\sigma_3 p(\vec{x})\rangle\varphi(\vec{x}) - \langle p^\dagger(\vec{x})\sigma_3 n(\vec{x})\rangle\varphi^*(\vec{x})], \quad (5)$$

where n and p are the neutron and proton field operators. One can derive an upper limit on the absolute magnitude of the space average of the quantity $\langle n^\dagger(\vec{x})\sigma_3 p(\vec{x})\rangle$ which is valid, independently of the nucleon mass, for any state of the system of nucleons:

$$|\langle\langle n^\dagger(\vec{x})\sigma_3 p(\vec{x})\rangle\rangle| \le \rho X^{1/2}(1-X)^{1/2}. \quad (6)$$

To prove (6) consider the field

$$\omega(\vec{x}) = X^{1/2}n(\vec{x}) - (1-X)^{1/2}\sigma_3 e^{i\varphi(\vec{x})}p(\vec{x}) \quad (7)$$

and note that

$$0 \le \langle\omega^\dagger(\vec{x})\omega(\vec{x})\rangle = X\langle n^\dagger n\rangle + (1-X)\langle p^\dagger p\rangle - X^{1/2}(1-X)^{1/2}\langle n^\dagger(\vec{x})\sigma_3 p(\vec{x})\rangle e^{i\varphi(x)} - X^{1/2}(1-X)^{1/2}\langle p^\dagger(\vec{x})\sigma_3 n(\vec{x})\rangle e^{-i\varphi(x)}. \quad (8)$$

Let us choose $\varphi(\vec{x}) = -\arg\langle n^\dagger(\vec{x})\sigma_3 p(\vec{x})\rangle$, whence

$$2X^{1/2}(1-X)^{1/2}|\langle\langle n^\dagger\sigma_3 p\rangle\rangle| \le (1-X)\langle p^\dagger p\rangle + X\langle n^\dagger n\rangle = 2X(1-X), \quad (9)$$

which gives Eq. (6). Substitution of (3), (4), and (6) into (5) gives

$$\langle H_\pi\rangle \ge \rho V\{(X+2Y)\omega_k - 2[(X+Y)^{1/2} + Y^{1/2}][1-X]^{1/2}m_\pi^{-1}\rho^{1/2}X^{1/2}\omega_k^{-1/2}fk\}, \quad (10)$$

and it is clear that for small ρ, k, or f the pionic energy cannot be negative. Since a second-order phase transition will be characterized by the vanishing of E_π, we have established that Eq. (1) with $\mu = \epsilon_n^F$ is not the correct critical condition.

It is easy to find the state in which the equality in (10) is realized, and this will be the exact ground state of the system. Consider a new field

$$U(\vec{x}) = (1-X)^{1/2}n(\vec{x}) - iX^{1/2}\sigma_3 e^{ikz}p(\vec{x}) = \sum_q U_q e^{i\vec{q}\cdot\vec{x}}V^{-1/2} \quad (11)$$

and construct the electrically neutral state[6]

$$|\Psi_0\rangle = \prod_{q_F} U_q^\dagger|0\rangle. \quad (12)$$

A direct calculation of $\langle\Psi_0|H|\Psi_0\rangle$ gives back (10) with the equality sign. This is the static limit of the condensation energy given in Ref. 3 and gives the critical density $\rho_c = m_\pi^2\omega_k^3(kf)^{-2}3^{-3/2}$.

The value of the π^- chemical potential μ_{π^-} at the critical point can be obtained directly from the energy formula by rewriting it at the critical density as an independent function of the numbers of the three kinds of particles,

$$E_\pi = \omega_k[N_+ + N_- - \frac{2}{3^{3/4}}(N_+^{1/2} + N_-^{1/2})N_p^{1/2}], \quad (13)$$

and differentiating to obtain

$$\mu_{\pi^-} = \partial E_\pi/\partial N_- = 3^{-1/2}\omega_k, \quad (14)$$

where in the last step I have used the threshold ratio of $N_p/N_{\pi^-} = 4\sqrt{3} - 6 = 0.928$ [as determined by imposing the constraint $N_+ = N_- - N_p$ and minimizing (13) with respect to N_p/N_-].

From (14) it is seen that μ_{π^-} is still finite and large in the static limit, contrary to the assertion of Ref. 1. Equation (1) is now satisfied when (2) is substituted, if $\omega = \mu = \mu_{\pi^-}$ is given by (14) and ρ by ρ_c.

In the π^0 condensation problem, in contrast, there is no parameter μ. Equation (1) with $\mu = 0$ gives the correct critical condition for weak π^0 fields. We are not allowed to take the static lim-

it in this case since $\Pi(0,k)$ as calculated in perturbation theory has energy denominators which go as M_N^{-1}. In the π^- case the denominators which arise all have a term μ in them, numerically considerably larger than nucleon kinetic energy terms for the physical nucleon mass, and this allows the taking of the static limit.

These differences reflect the very different nature of the nuclear wave function in the two cases. In the π^- case the nucleon density remains uniform and the density-density correlation function remains the same as in the case of no pion condensation. As shown in Ref. 3 this is not only true in the free Fermi gas but also in the presence of any nuclear forces which are spin and isospin independent. Thus the force which is most responsible for determining the nuclear wave function, the hard core, can be taken into account completely in the π^- case.

In the π^0 case the opposite is true. When nuclear forces are left out the neutrons rearrange themselves extensively in order to minimize the interaction energy with the π^0 wave. The π^0 wave is necessarily a standing wave (since the π^0 field is self-conjugate) and the neutrons are inclined to move to where the pions are concentrated. The fact that the second-order perturbation calculation gives an interaction energy proportional to M_N is imply a reflection of the fact that the neutrons can be rearranged very easily, since they are so heavy. In this case nuclear forces can be taken somewhat into account, e.g., by the use of effective mass. But the effect of the nuclear wave function, in particular of the correlations due to the hard core, has not been taken into account in Ref. 1 in any way which is nearly as conclusive as in the π^- case.

There is another important respect in which the work on the π^- case reported in Ref. 3 is more complete than the work on π^0 reported in Ref. 1. This is in the pionic corrections due to all of the other modes of the pion field except for the condensed mode. These were evaluated to order f^2 for the π^- case in Ref. 3.[7] They cost a good fraction of the condensation energy and raised the critical density a great deal. The analogous calculation has not been carried out for the π^0 case. An important point brought out in the π^- case was that these corrections are *not given correctly* by taking the expectation value of the one-pion-exchange potential in the state of the nucleons described above. There is in addition a large term of order f^2 which comes from the change in the ordinary self-energy part $N \to N + \pi \to N$ when the nucleon is immersed in the pion-condensed medium.

I do not wish to use the above points to argue that π^- condensation will occur and π^0 will not. Neither case has been analyzed sufficiently to allow a conclusion. My point is that many aspects of the problem have been analyzed more thoroughly in the π^- case than in the π^0 case.

There is, furthermore, a serious doubt about what a π^0 condensation really means. Once the translational invariance is broken in the standing-π^0-wave case, the system is a sort of one-dimensional crystal with the neutron spins alternating on successive planes such as to lower the energy through the spin dependence of the one-pion-exchange potential. One can call the wave of pion field generated by these sources a condensed π^0 mode if one wants to. But is this state really preferred to a three-dimensional crystalline structure with spins alternating on nearest neighbors?[8]

It should be noted that in the neutron-star problem one is at least as interested in the properties of the matter at densities considerably above the critical density as one is in the region just above onset. There is also the possibility that when all of the effects have been put in, the transition will be of the first class. For either of these problems the Migdal approach to the π^0 case is of no help. The perturbation theory in the strength of the condensed field fails for pion concentrations which are quite small (see Ref. 3). For strong π^0 fields the correct calculation is a simple one-dimensional band-structure calculation which could be done numerically (analogous to the standing-wave π^- calculation given in Ref. 8, but simpler).

One thing which can be proved analytically in the case of a π^0 condensation in an otherwise free-neutron gas is that a $\pi^0\pi^0$ repulsion is not necessary to stabilize the system. This is in contrast to the view expressed by Migdal[9] and by Barshay, Vagradov, and Brown.[10] The point is that for small values of $\varphi_{\pi 0}$ the potential-energy term and the free-pion energy term go as $\varphi_{\pi 0}^2$ (i.e., like N_π), and it looks as though a collapse might occur when the coefficient is negative. However, for large $\varphi_{\pi 0}$ the negative potential term goes like $\varphi_{\pi 0}$ (or $\sqrt{N_\pi}$) while the free part of the energy still goes as $N_\pi \omega_k$.[11] Thus the number of π^0's saturates. This is another illustration of the necessity of solving the problem with strong pion fields as was done for the π^- case in Ref. 3 and has yet to be done for the π^0 case.

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[1]A. B. Migdal, Phys. Rev. Lett. 31, 257 (1973).

[2]That the lowest-energy state involves a small admixture of π^+'s was noted by R. F. Sawyer and A. C. Yao [Phys. Rev. D 7, 1579 (1973)]. The π^+ mixture also is the answer to the question of how the current gets canceled in the ground state, as demanded by Baym's general result [Phys. Rev. Lett. 30, 1340 (1973)]. If one does restrict to π^- particles, there is a slightly different condition, analogous to (1), which gives the critical density given by R. F. Sawyer, Phys. Rev. Lett. 29, 382 (1972); D. J. Scalapino, Phys. Rev. Lett. 29, 386 (1972); and R. F. Sawyer and D. J. Scalapino, Phys. Rev. D 7, 953 (1973). I am indebted to George Bertsch for showing me the derivation of this condition and for emphasizing that π^+'s are automatically included in (1).

[3]Sawyer and Yao, Ref. 2.

[4]Sawyer, Ref. 2; Scalapino, Ref. 2; Sawyer and Scalapino, Ref. 2.

[5]Because both notations are standard I have used the symbol Π for both the pion canonical momentum operator $\Pi(\vec{x})$ and for the pion proper polarization part $\Pi(\omega, \vec{k})$.

[6]The average proton density calculated from the state (12) is $X\rho$. There are fluctuations of course, but they do not give rise to finite Coulomb energy per unit volume in the $V \to \infty$ limit. Any set of "U particle" modes other than the filled Fermi sphere of (12) will give a degenerate state in the static limit. When the nucleon mass is large but finite, however, the minimum energy solution will be given by (12).

[7]Hard-core correlations were incompletely taken into account in this calculation, however. A formal approach has been developed by G. Bertsch and M. Johnson (to be published) in which the remaining hard-core corrections can be easily taken into account in determination of the critical density.

[8]See V. Canuto and S. M. Chitre, Phys. Rev. Lett. 30, 999 (1973); L. Nosanow and L. Parish, in Proceedings of the Sixth Texas Symposium on Relativistic Astrophysics, New York, December 1972 (New York Academy of Sciences, to be published).

[9]A. B. Migdal, Zh. Eksp. Teor. Fiz. 63, 1993 (1972) [Sov. Phys. JETP (to be published)].

[10]S. Barshay, G. Vagradov, and G. Brown, to be published.

[11]This can be worked out using the methods of Ref. 3.

Vacuum Polarization in High-Z, Finite-Size Nuclei*

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We have calculated vacuum-polarization charge densities to all orders $(Z\alpha)^n\alpha$ for realistic nuclei of finite extent. In muonic Pb, orders $n \geq 3$ contribute -34 ± 2 eV to the $5g \to 4f$ transitions, compared with the point nucleus value of -50 eV. This *increases* the discrepancy (first reported by Dixit *et al.*) to $E(\text{theor}) - E(\text{expt}) = 57 \pm 18$ eV ($5g_{9/2} \to 4f_{7/2}$) and 72 ± 21 eV ($5g_{7/2} \to 4f_{5/2}$). Contributions to low-lying states show nearly an order-of magnitude reduction.

A variety of recent and planned experiments have spurred renewed interest in the problem of vacuum polarization (VP). Most prominent among these is the measurement of energies of high transitions in high-Z muonic atoms, as reported by Dixit *et al.*[1] and also by Backenstoss *et al.*[2] and Walter *et al.*[3] Two of these particular experiments, which involved very high precision (50 ppm), exhibit a persistent discrepancy of more than 2 standard deviations from theory. The theory requires a number of delicate considerations. One of the most uncertain theoretical contributions has been the higher-order ($n \geq 3$) VP, which was evaluated for a point nucleus. The work reported here shows the inadequacy of such an approximation even though the relevant muonic orbits are well outside the nuclear radius.

Other muonic-atom studies require the calculation of higher-order VP effects. As an example, transitions involving the $1s$ state are claimed to measure the nuclear polarization, but previous calculations show that higher-order VP effects