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Spectra of Strong Langmuir Turbulence*

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Strong Langmuir turbulence is described in terms of a random set of blobs of self-trapped plasma waves. The interaction of these blobs leads to the generation of power spectra $\langle |E_k|^2\rangle \propto k^{-2}$ that agree with the results of one-dimensional computer simulation.

The strongly turbulent regime is of great importance in the heating of plasma by high-current relativistic electron beams or by powerful lasers. Indeed, in both cases an important process by which the energy of the beam or the transverse electromagnetic wave is converted to plasma energy is via Langmuir oscillations. If the input power and pulse duration are large enough the energy density of oscillations may become very high. The applicability of weak-turbulence theory is restricted by the condition

$$W/nT < (k\lambda_D)^2, \tag{1}$$

where W is the energy density of the oscillations, nT is the thermal energy density, k is the typical wave number of the oscillations, and λ_D is the Debye length. If this condition is not satisfied, the characteristic rates of nonlinear interactions $\delta\omega \sim \omega_p(W/nT)$ become greater than the frequency spread due to thermal effects $\delta\omega_k \sim \omega_p(k\lambda_D)^2$; $\omega_p = (4\pi ne^2/m)^{1/2}$ is the electron plasma frequency. It has been shown[1] that the Langmuir spectrum is unstable with respect to low-frequency density perturbations when $W/nT > (\Delta k\lambda_D)^2$, where Δk is the width in the k spectrum. For the case $\Delta k/k \ll 1$, this instability is identified with the decay instability or at higher amplitudes with the oscillating two-stream instability. In the opposite limit $\Delta k \sim k$, when the resonant conditions cannot be satisfied for the entire set of k in the spectrum, only the modulational instability of Ref. 1 can exist. Thus even when $W/nT \ll 1$, strongly correlated states may be a feature of Langmuir turbulence. There exist, then, the problems of the dynamics of such a turbulent state, of the dissipation of wave energy, of beam-plasma and laser-plasma interactions in this regime, etc.

Processes of this kind were investigated in some detail by Zakharov.[2] He suggested that strong Langmuir turbulence should be described not in the Fourier representation but in terms of strong nonlinear waves or "solitons." The strongly turbulent state is likely to consist of a set of steady or quasisteady "blobs" localized in space with random positions. A one-dimensional (1D) example of such a theory is given by Rudakov.[3] The basic equations describing Langmuir turbulence are[2]

$$\nabla\cdot\left(i\frac{\partial}{\partial t}\vec{E}+\tfrac{3}{2}\omega_p\lambda_D^2\nabla\nabla\cdot\vec{E}\right)=\tfrac{1}{2}\omega_p\nabla\cdot\left(\frac{\delta n}{n_0}\vec{E}\right) \qquad (2)$$

and

$$\frac{\partial^2}{\partial t^2}\delta n-c_s^2\nabla^2\delta n=\frac{1}{8\pi M}\nabla^2|E|^2, \qquad (2')$$

where δn is the low-frequency plasma density perturbation, $\vec{E}$ is the complex amplitude of the high-frequency electric field, and $c_s=(T_e+\gamma T_i/M)^{1/2}$ is the sound velocity. In a 1D system a stationary solution of Eqs. (2) and (2′) for $v_g<c_s$ which has the form of Langmuir solitons is given by[3]

$$E(x,t)=E_m\sin(kx-\omega t)/\cosh(k_0\xi);$$
$$\xi=x-v_gt,\quad v_g=3k\lambda_D^2\omega_p, \qquad (3)$$
$$k_0\lambda_D=E_m/(48\pi nT)^{1/2},$$

and

$$\omega=\omega_p(1+\tfrac{3}{2}k^2\lambda_D^2-E_m^2/32\pi nT).$$

Its amplitude E_m and propagation velocity v_g determined by k are independent of each other. The energy of each soliton is

$$\mathcal{E}_s=\int_{-\infty}^{\infty}d\xi\,E^2/4\pi=(48\pi nT)^{1/2}\lambda_DE_m/2\pi. \qquad (4)$$

The Fourier expansion of the soliton (3) is

$$E(x,t)=\sin(kx-\omega t)\int_0^\infty dk'\cos(k'\xi)E_{k'},$$

with

$$E_k'=E_m/k_0\cosh(\tfrac{1}{2}\pi k'/k_0). \qquad (5)$$

At the present time a number of 1D computer simulation studies have been performed in the strongly turbulent regime.[4-8] It is interesting to compare the turbulent spectra $\langle|E_k|\rangle^2\propto k^{-2}$ obtained in Refs. 6–8 with the consequences of the concept of a random set of solitons described in Ref. 3.

Let us construct a 1D model in which we have a set of N solitons in a length L where L can be so chosen that within L most of the solitons have roughly the same amplitude but with arbitrary positions in space. If the characteristic time for the energy density fluctuation δW is much greater than $(\omega_pW/nT)^{-1}$ we may postulate that $W=\text{const}$. Thus the soliton amplitude E_m depends upon the number of solitons N through

$$WL=N\mathcal{E}_s=(48\pi nT)^{1/2}\lambda_DNE_m(N)/2\pi. \qquad (6)$$

The characteristic soliton scale length k_0^{-1} turns out to be

$$k_0\lambda_D=\tfrac{1}{24}(W/nT)(L/\lambda_D)N^{-1}. \qquad (7)$$

The maximum number of solitons in L is restricted by the condition of close packing:

$$N\sim k_0L,$$

i.e.,

$$N_{\max}\approx(1/2\sqrt{6})(W/nT)^{1/2}(L/\lambda_D). \qquad (8)$$

If $N\ll N_{\max}$ (rarefied packing) the Fourier components of the spectrum of each soliton can be written as

$$E_k(N)\approx E_m(N)\{k_0(N)\cosh[\tfrac{1}{2}\pi k/k_0(N)]\}^{-1}. \qquad (9)$$

Let us introduce the probability $P(N)$ for the system to be in the N-soliton state and furthermore regard these N solitons to have random phases and positions. The Fourier power spectrum of the entire N-soliton state is given by

$$\langle|E_k|^2\rangle=\frac{12T^2}{Le^2}\int_{N_{\min}}^{N_{\max}}dN\,NP(N)\times\left[\cosh\left(3\frac{T^2}{e^2}\frac{kN}{WL}\right)\right]^{-2}. \qquad (10)$$

Note that if the N solitons had coherent phases and positions the factor N in the integrand would be replaced by N^2. We may replace the $(\cosh x)^{-2}$ term by a step function which cuts off the integration at $N_0=\frac{1}{3}(e^2/T^2)(WL/k)$. Thus we obtain

$$\langle|E_k|^2\rangle=(12T^2/Le^2)\int_{N_{\min}}^{N_0}dN\,NP(N). \qquad (11)$$

It is interesting to note that we obtain just the result of the numerical simulations if we take $P(N)=\text{const}$, which gives

$$\langle|E_k|^2\rangle\sim k^{-2}\text{ for }k>eW^{1/2}/T. \qquad (12)$$

If this spectrum is introduced in the quasilinear velocity diffusion coefficient $D(v)$, then we obtain $D(v)\propto v$. Before discussing the reasons for this particular choice of $P(N)$ we give a brief consideration of the 3D problem. From Eq. (2) the same dependence, viz. $k_0\lambda_D\propto E_{\max}/(nT)^{1/2}$, is obtained as in the 1D case. But as of this time we do not know of any stationary solution of the system (2) and (2′). However the preceding results have been derived using only the correlation between the amplitude and space scales of the solitons. We may therefore extend these calculations to 3D where, strictly speaking, solitons may not exist, but blobs having only the same characteristic scales as solitons are present. Thus we establish the following scale dependences:

$$\mathcal{E}_s \sim E_m^{\,2} k_0^{\,-3} \sim E_m^{\,-1}, \quad WL^3 = N\mathcal{E}_s \propto NE_m^{\,-1}, \qquad k_0 \propto E_m \propto N. \tag{13}$$

This means that in a 3D problem the condition of "close packing" determines $N_{\min} \propto W^{3/2}L^3$. The Fourier components of a blob of scale size k_0 is therefore estimated to be $E_{\vec{k}} \propto E_m k_0^{\,-3}/F(k/k_0)$, where $F(k/k_0)$ is approximated by a step function with cutoff at $k \approx k_0$. Finally the power spectrum of the N-soliton state obtained following the procedure outlined earlier is

$$\langle |E_{\vec{k}}|^2\rangle \propto \int_{WL^3k > N_{\min}}^{N_{\max}} dN\, N^{-3} P(N). \tag{14}$$

It is very interesting to note that the flat distribution $P(N) = \text{const}$ gives the same spectrum $\langle |E_{\vec{k}}|^2\rangle \propto k^{-2}$ as for the 1D case.

We now offer some arguments in support of the choice of $P(N)$ used above in the limit $k_0\lambda_D < (m/M)^{1/2}$. In the 1D case the decay of one into two solitons is impossible but the conservation of energy admits the coalescence of two solitons into one. The characteristic time scale $\nu_{\text{eff}}^{\,-1}$ for such a process is roughly $L/Nv_g(k_0)$ which gives $\nu_{\text{eff}} \approx \omega_e(W/nT)$ which is independent of N. Hence a Liouville-type equation

$$\partial P/\partial t = \partial[\nu_{\text{eff}} P(N)]/\partial N \tag{15}$$

in steady state furnishes $P(N) = \text{const}$ for the main part of the spectrum if the energy source excites the long scales.

On the contrary, in a 3D situation the generation of short scales is possible through the decay of blobs. Such decay is induced by collisions between blobs. We do not take into consideration the spontaneous decay processes as we have assumed that blobs of this size are quasistable. The effective frequency for the generation of short-scale-length blobs can be estimated as

$$\nu_{\text{eff}} = \sigma n v \sim k_0^{\,-2} N v_g(k_0) \sim N k_0^{\,-1} = \text{const}$$

from (13). Hence Eq. (15) gives $P(N) = \text{const}$ again.

The estimates given above are valid only if $v_g(k_0) < c_s$. In the opposite case solitons can exist only under the condition $v_g < c_s < v_g(k_0)$ and their dynamics is rather complicated because interaction between solitons can be accompanied by the radiation of sound. The basis for a "flat" distribution therefore needs further investigation.

Thus, under the conditions of strong correlation (1), a new turbulent state consisting of a random set of blobs can develop in which each blob is composed of self-trapped Langmuir waves. The interaction of blobs leads to the establishment of a spectrum $\langle E_k^{\,2}\rangle \sim k^{-2}$ and a flux of energy takes place from the source region $k \sim k_{\text{unstable}}$ to the short-wavelength region $k \sim \lambda_D^{\,-1}$ where wave energy is absorbed by the plasma. In the region of strong Landau damping $k\lambda_D \geq 1$ we obviously expect the spectrum to fall off much faster than k^{-2}.

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