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Generalized Thermodynamic Potential for the Convection Instability

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A new generalized thermodynamic potential is presented which governs the stability, the dynamics, and the fluctuations of a liquid fluid layer near the Benard point, even in the nonlinear domain. By analogy, a large number of results obtained for other systems are shown to be applicable to the Benard problem.

The Benard convection instability[1] of a plane horizontal fluid layer heated from below has received a great amount of interest for several reasons. In the first place it is probably the simplest nontrivial example of an instability in fluid dynamics. By the same token, the appearance of convection cells at the Benard point is also a very clear-cut example of a "dissipative structure."[2] Such structures are well known to appear in many systems if they are driven by some external force into a nonlinear domain far from thermal equilibrium. The very abrupt appearance of dissipative structures, like convection cells, at remarkably well-defined values of those parameters which characterize how far away the system is from thermal equilibrium (the Rayleigh number R in our case) has sometimes been compared to phase transitions in equilibrium systems. This comparison is even more intriguing, since the appearance of a dissipative structure, like a phase transition, is accompanied by an obvious change of the symmetry of the macroscopic state of the system. However, so far the validity of such a comparison has not yet been checked. More specifically it has not yet been established whether the order introduced at the Benard point by the appearance of convection cells is a long-range order in the usual sense or rather an order of finite range.

It has been difficult so far to decide this question, because of the lack of a sufficiently general thermodynamic potential which could describe simultaneously the fluctuations, the dynamics, and the stability near the Benard point in a similar way as the free energy does for thermal-equilibrium systems. Attempts to construct such a potential have been numerous. The most remarkable and general approach up to now has been made by Glansdorff and Prigogine.[2] However, the generalized thermodynamic potential they constructed, the "excess entropy production" or a generalization thereof, although being a very useful and rather easily constructed quantity, governs the dynamics and the stability only in the linear vicinity of a stationary nonequilibrium state, while a nonlinear analysis is necessary near an instability. In the present note, I would like to present a new generalized thermodynamic potential for the Benard problem.[3]

Let us start from the hydrodynamic equations of the liquid within the conventional Boussinesq approximation[1] and add the random driving forces due to thermodynamic fluctuations,[4,5] thereby assuming local thermodynamic equilibrium. As has been shown,[6] for $R < R_c$ the motionless heat-conduction state is the only stable solution, while for a certain range of $R > R_c$ that state becomes unstable and two-dimensional rolls (i.e., independent of the y direction parallel to the fluid layer) are stable within a certain range of wave numbers.[7] Hence, for the sake of brevity, the discussion will be restricted to nearly two-dimensional flows.

Employing perturbation techniques described by Newell and Whitehead,[8] but keeping in addition fluctuating force terms, a Fokker-Planck functional equation is derived from the Boussinesq-Langevin equations. It is satisfied by the probability density W for observing a certain complex amplitude $w(x,y)$ of the (vertical) z component of the velocity field $v_z(x,y,z)$,[9]

$$v_z(x,y,z)=[w(x,y)\exp(ik_cx)+\text{c.c.}]\sin\pi z. \quad (1)$$

For simplicity I have assumed free boundary conditions ($v_z=0$) on top and on the bottom of the liquid layer, which leads to a critical Rayleigh number $R_c=27\pi^4/4$, and to a critical wave number $k_c=\pi/\sqrt{2}$.[1] Furthermore, I have assumed that the z dependence is completely fixed (because of the boundary conditions) and consider only the lowest order vertical mode. In this derivation of the Fokker-Planck equation I have to assume in addition that the fluctuations of $w(x,y,t)$ are slowly varying functions of space and time in the vicinity of the Benard point (over distances and times of the order of 1), and that a power-series expansion of the velocity field in powers of $(R-R_c)^{1/2}$ is possible. Both assumptions are consistent with the results obtained by making them. The equation then takes the form

$$\frac{\partial W}{\partial t}=\frac{Q}{1+P}\iint_{(F)}dx\,dy\left[\frac{\delta}{\delta w(x,y)}\left(\frac{\delta\varphi}{\delta w^*(x,y)}+\frac{\delta}{\delta w^*(x,y)}\right)W+\text{c.c.}\right] \quad (2)$$

with

$$Q=\pi^2(Q_1+Q_2)/(1+P).$$

Q_1 and Q_2 are the intensities of the driving stochastic heat flux and momentum flux densities, respectively, which in the units used here[9] are given by

$$Q_1=g\beta KT^2/\delta C_p\nu^2\,\Delta T,\quad Q_2=KT/\delta\nu^2 l.$$

φ, in Eq. (2), is the following functional of $w(x,y)$:

$$\varphi=Q^{-1}\iint_{(F)}dx\,dy\left[-\tfrac{3}{2}\pi^2\epsilon|w|^2+\tfrac{1}{4}P^2|w|^4+4|\partial w/\partial x-(i/\sqrt{2}\pi)\partial^2w/\partial y^2|^2\right]. \quad (3)$$

Obviously, $W\sim\exp(-\varphi)$ is a time-independent solution of Eq. (2).[10] Hence, Eq. (3) is the central result. It combines the same amount of information on the dynamics, stability, and fluctuations as the free energy does for equilibrium systems. From Eq. (2) one can immediately derive the linear force flux relationship for the averages,

$$\langle\dot{W}\rangle=-[Q/(1+P)]\langle\delta\varphi/\delta w^*\rangle. \quad (4)$$

Indeed, dropping the averages in Eq. (4) leads directly back to the hydrodynamic equations derived in Ref. 8 for the present case. Therefore, it becomes at least plausible that Eqs. (2) and (3) hold within the same approximations, if in addition fluctuating force terms are taken into account. The details of my derivation will be presented elsewhere.

If w is governed by the hydrodynamic equation associated to Eq. (4), φ can only decrease in time, since the coefficient in Eq. (4) is negative. Hence, states which minimize φ are stable and occur with maximum probability. The state which absolutely minimizes φ for $R>R_c$ is the uniform state $|w|^2=3\pi^2\epsilon/P^2$ which by Eq. (1) corresponds to a completely regular alignment of two-dimensional rolls.

Other extrema of φ are

$$w_k=f_k\exp[i(k-k_c)x],$$

$$|f_k|^2=[3\pi^2\epsilon-8(k-k_c)^2]/P^2.$$

Their value of φ is larger than the φ value of the state $k=k_c$. For $k<k_c$, $k-k_c>\pi\sqrt{\epsilon}/2\sqrt{2}$, these states are in any case unstable with respect to oblique modes and sideband generation in x direction, respectively,[6,8] as can be shown by an analysis of φ in the vicinity of these states. However, even in the stability region $0\leq k-k_c<\pi\sqrt{\epsilon}/2\sqrt{2}$ these states are subject to a decay towards the (most probable) state $k=k_c$. In such a decay process, the velocity field has to fluctuate from the local minimum of φ, given by $w_k(x)$, across a barrier of larger φ to the next smaller k value allowed by the boundary conditions. The most probable path in phase space the system can chose for this fluctuation is via a saddle point of φ, again given by $\delta\varphi/\delta w^* = 0$. In fact, this problem has already been considered and solved explicitly[11] in connection with current fluctuations in one-dimensional superconductors. The entire formal analysis may immediately be taken over. Among the results which become available are analytic expressions for the local perturbations at the saddle points of φ and formulas giving the height of the φ barrier. For the sake of brevity I give these results only in the limit $k \to k_c$, where the saddle point of φ is given by

$$|w|^2 = (3\pi^2\epsilon/P^2)\{\tanh[x(3\pi^2\epsilon)^{1/2}/4]\}^2$$

and has the height

$$\Delta\varphi = 8\sqrt{3}\pi^3 L_y \epsilon^{3/2}/QP^2.$$

L_y is the total cell length in the y direction. For most liquids $Q_1 + Q_2$ is very small and one would have to choose R extremely close to R_c in order to observe the jumps across the barrier of φ for $R > R_c$ (i.e., the sudden disappearance of one roll in the roll pattern). Hence, rolls within the stability region $k > k_c$ will have a very long relaxation time.[10]

Next, we consider the correlation length in the ordered state of the layer. Equation (3), interpreted as a free energy, allows immediately the statement that true long-range order cannot be present for $R > R_c$. Spatial fluctuations in the y direction enter φ only by the second-order derivative, and are hence very easily excited at long wavelengths. The first-order derivative in the x direction, entering Eq. (3), is also not sufficient to ensure long-range order, since it still allows for a free spatial diffusion (rather than oscillation) of the phase of the complex velocity field with a finite correlation length. By Eq. (1), this is synonymous with a finite correlation length of the relative position of the rolls and is thus not reconcilable with long-range order. These general arguments will apply also to other conceivable cell forms. While long-range order is thereby ruled out in principle, the question of how long the range really is has still to be answered separately. This is easy for rolls which are homogeneous in the y direction ($\partial/\partial y = 0$). In that case, Eq. (3) reduces to the well-known free-energy expression of one-dimensional Ginzburg-Landau fields, which have recently been analyzed in great detail.[12] Among the results which are readily available from these numerical studies are the correlation lengths ξ_1 and ξ_2 of the field amplitude w and its absolute square, respectively, as well as the expectation value $\langle |w|^2 \rangle$, all as a function of $\epsilon = (R - R_c)/R_c$. The results show a smooth transition region around $\epsilon = 0$ of width $\Delta\epsilon = (P^2Q/L_y)^{2/3}/3\pi^2$. ξ_1 and ξ_2 both have the same order of magnitude, $4(L_y/P^2Q)^{1/3}$, in that region. ξ_2 has a maximum of about half that value for $\epsilon \simeq \Delta\epsilon$ and is $\xi_2 = (2/3\pi^2|\epsilon|)^{1/2}$ outside the transition region. ξ_1 increases rapidly through the transition and remains large for $R > R_c$. I want to point out that the transition region I obtain is extremely narrow for realistic fluids [$\Delta\epsilon \simeq 3 \times 10^{-7}(L_y/l)^{-2/3}$ for water at $T = 20°C$] and that the correlation lengths are extremely long in that region [$\xi_{1,2} \simeq 1 \times 10^3 (L_y l^2)^{1/3}$, for the same example] so that the transition can be considered to be sharp in very good approximation.

The counterparts in the time domain of the two correlation lengths are the two correlation times τ_1 and τ_2 of w and $|w|^2$, respectively, which show a very similar behavior. They can be estimated by neglecting the spatial derivative terms in φ (upon the assumption that the coherence lengths are much larger than the sample lengths) and solving the dynamical problem posed by Eq. (2). Again, this has already been accomplished before, this time in laser theory.[13] The two correlation times within the transition region [now having the width $\Delta\epsilon = (Q/F)^{1/2}P/3\pi^2$] have the order of magnitude $(F/Q)^{1/2}2(1+P)/P$; τ_2 has a maximum of about $\frac{1}{5}$ that value for $\epsilon \simeq 2\Delta\epsilon$ and is $\tau_2 = (1+P)/3\pi^2|\epsilon|$ outside the transition region, indicating a slowing down of entropy or energy fluctuations with $k = k_c$ in the transition region. τ_1 is about $(F/Q)^{1/2}(1+P)/P$ for $R = R_c$; it increases monotonically through the transition and is $\tau_1 = 6\pi^2(1+P)\epsilon F/P^2Q$ for $R > R_c$ outside the transition region. Again the numbers for realistic fluids show the transition to be sharp from an experimental point of view.

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[1]Cf., e.g., S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, England, 1961).

[2]P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability, and Fluctuations* (Interscience, New York, 1971).

[3]This potential is a special example of a class of potentials first discussed by R. Graham and H. Haken, Z. Phys. 243, 289, (1971), and 245, 141 (1971); see also R. Graham, in *Springer Tracts in Modern Physics, Ergebnisse der exakten Naturwissenschafter*, edited by G. Höhler (Springer, Berlin, 1973), Vol. 66, p. 1. A

comparison of these potentials with the excess entropy production has been given by R. Graham, in *Coherence and Quantum Optics*, edited by L. Mandel and E. Wolf (Plenum, New York, 1973), p. 851.

[4]L. D. Landau and E. M. Lifshitz, *Fluid Dynamics* (Pergamon, New York, 1959).

[5]These equations were applied to the Benard problem before in K. M. Zaĭtsev and M. I. Shliomis, Zh. Eksp. Teor. Fiz. 59, 1583 (1970) [Sov. Phys. JETP 32, 866 (1971)], where a linearized theory of the transition has been given.

[6]A. Schlüter, D. Lortz, and F. Busse, J. Fluid Mech. 23, 129 (1965).

[7]The well-known hexagonal convection cells would require non-Boussinesq terms in the equations of motion; cf. R. E. Krishnamurti, J. Fluid Mech. 33, 445 (1968).

[8]A. C. Newell and J. A. Whitehead, J. Fluid Mech. 38, 279 (1969).

[9]Length, time, velocity, and temperature are measured in units of l, l^2/ν, ν/l, and $(\Delta T\nu^3/g\beta\kappa l^3)^{1/2}$, respectively. l, ν, ΔT, β, g, and κ are the cell thickness, the kinematic viscosity, the externally maintained temperature difference between bottom and top of the layer, the fluid's volume expansion coefficient, the gravitational acceleration, and the thermometric conductivity, respectively. The Rayleigh number $R = g\beta\Delta T\, l^3/\nu\kappa$ and the Prandtl number $P=\nu/\kappa$ are formed from these constants. Later we will have to use the fluid density ρ, the average fluid temperature T, its specific heat C_p, the cell length in the y direction (L_y), and the total area F of the layer. K is Boltzmann's constant. I abbreviate $(R-R_c)/R_c=\epsilon$.

[10]The second-order functional derivatives taken at the same point, which appear in Eq. (2), are not well defined if operating on a functional containing $|w|^2$, as, e.g., the functional φ given by Eq. (2). This difficulty suggests the use of a certain microscopic length as a cutoff in order to delocalize the second-order functional derivatives. The time-independent solution of Eq. (2) is then found to be cutoff independent, while fluctuation rates which are determined from Eq. (2) will depend on the cutoff.

[11]J. S. Langer and V. Ambegaokar, Phys. Rev. 164, 498 (1967).

[12]D. J. Scalapino, M. Sears, and R. A. Ferrell, Phys. Rev. B 6, 3409 (1972).

[13]H. Risken, in *Progress in Optics*, edited by E. Wolf (North-Holland, Amsterdam, 1970), Vol. 8; H. Haken, in *Handbuch der Physik*, edited by S. Flügge (Springer, Berlin, 1970), Vol. 25.

Spectra of Strong Langmuir Turbulence*

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Strong Langmuir turbulence is described in terms of a random set of blobs of self-trapped plasma waves. The interaction of these blobs leads to the generation of power spectra $\langle|E_k|^2\rangle \propto k^{-2}$ that agree with the results of one-dimensional computer simulation.

The strongly turbulent regime is of great importance in the heating of plasma by high-current relativistic electron beams or by powerful lasers. Indeed, in both cases an important process by which the energy of the beam or the transverse electromagnetic wave is converted to plasma energy is via Langmuir oscillations. If the input power and pulse duration are large enough the energy density of oscillations may become very high. The applicability of weak-turbulence theory is restricted by the condition

$$W/nT < (k\lambda_D)^2, \tag{1}$$

where W is the energy density of the oscillations, nT is the thermal energy density, k is the typical wave number of the oscillations, and λ_D is the Debye length. If this condition is not satisfied, the characteristic rates of nonlinear interactions $\delta\omega \sim \omega_p(W/nT)$ become greater than the frequency spread due to thermal effects $\delta\omega_k \sim \omega_p(k\lambda_D)^2$; $\omega_p = (4\pi ne^2/m)^{1/2}$ is the electron plasma frequency. It has been shown[1] that the Langmuir spectrum is unstable with respect to low-frequency density perturbations when $W/nT > (\Delta k\lambda_D)^2$, where Δk is the width in the k spectrum. For the case $\Delta k/k \ll 1$, this instability is identified with the decay instability or at higher amplitudes with the oscillating two-stream instability. In the opposite limit $\Delta k \sim k$, when the resonant conditions cannot be satisfied for the entire set of k in the spectrum, only the modulational instability of Ref. 1 can exist. Thus even when $W/nT \ll 1$, strongly correlated states may be a feature of Langmuir turbulence. There exist, then, the problems of the dynamics of such a turbulent state, of the dissipation of wave energy, of beam-plasma and laser-plasma interactions in this regime, etc.

Processes of this kind were investigated in