Zero-Field Susceptibility of the Two-Dimensional Ising Model near T_c

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We compute exactly the four coefficients $C_{0\pm}$ and $C_{1\pm}$ in the expansion

$$\beta^{-1}\chi = C_{0\pm}|1-T_c/T|^{-7/4} + C_{1\pm}|1-T_c/T|^{-3/4} + O(1)$$

for the susceptibility of the rectangular two-dimensional Ising model.

In the absence of a magnetic field H, the magnetic susceptibility of the two-dimensional ferromagnetic Ising model is unbounded at the critical temperature T_c. In this Letter we calculate *exactly* all the divergent terms near T_c, both for $T > T_c$ and for $T < T_c$.

Nearly three decades ago, Onsager[1] gave his celebrated treatment of the two-dimensional Ising model when $H = 0$, and in 1952 Yang[2] calculated exactly the spontaneous magnetization. Magnetic susceptibility is thus the natural quantity to study next.

Consider a rectangular Ising model with pair energies E_1 and E_2. We define the standard variables

$$z_1 = \tanh\beta E_1,$$

$$z_2 = \tanh\beta E_2,$$

$$\alpha_1 = z_1(1-z_2)(1+z_2)^{-1},$$

$$\alpha_2 = z_1^{-1}(1-z_2)(1+z_2)^{-1},$$

and the on-line correlation $S_N = \langle\sigma_{00}\sigma_{0N}\rangle$. For large N and *fixed* temperature T, the asymptotic behavior of S_N has been studied systematically[3] by using Wiener-Hopf sum equations.

For $T \neq T_c$, this analysis has also been extended[4] to the off-line correlation $\langle\sigma_{00}\sigma_{MN}\rangle$. If we take the explicit formulas (4.22) and (3.31) of Ref. 4, we observe that "rotational symmetry" obtains for $T \to T_c$. More precisely, if we define

$$R^2 = M^2[z_2(1-z_1^2)]^{-2} + N^2[z_1(1-z_2^2)]^{-2},$$

then in the limit $T \to T_c$, $R \to \infty$ such that $R|1-z_1-z_2-z_1z_2| = t$ is fixed, we find that $\langle\sigma_{00}\sigma_{MN}\rangle$ of both (4.22) and (3.31) is of the form, for any $\epsilon > 0$,

$$R^{-1/4}F_1(t) + R^{-5/4}F_2(t) + O(R^{-9/4+\epsilon}), \tag{1}$$

where F_2 is F_1 times an elementary function of z_1 and z_2. The important point is the following: If the Wiener-Hopf sum equations are iterated more times, then to each order the form (1) still holds. The question of convergence of this iteration has been studied by Fisher and Hartwig.[5] Thus, to the first two orders, the spin correlation function depends only on R but not on N and M separately. It is therefore sufficient to concentrate on S_N.

Although this consideration is sufficient to establish "rotational symmetry," it is still necessary to study in more detail the desired limit of $N \to \infty$, $T \to T_c$ such that $N|T-T_c|$ is fixed. For this purpose we again define[3] $x_0(N) = S_{N-1}/S_N$, which can be determined by solving a set of simultaneous linear equations. In the limit of interest, we can replace this set of linear equations by integral equations, after overcoming a number of technical difficulties. In this way, $x_0(N)$ and hence S_N can be determined to the desired accuracy as

$$x_0 \sim \max(1,\alpha_2) + |\ln\alpha_2| \lim_{Z\to\infty}\{-Z^{1/2}(Z+1)^{1/2} - \pi^{3/2}\lim_{s\to L}[\bar{x}(s,Z)/\tilde{x}(s)]\lim_{s\to 0}[s^{1/2}\tilde{x}(s)] + \pi^{3/2}\lim_{s\to 0}[s^{1/2}\bar{x}(s,Z)]\}, \tag{2}$$

where $\tilde{x}(s)$ and $\bar{x}(s,Z)$ are, respectively, solutions of the equations[6]

$$\int_0^L ds' K_0(|s-s'|)\tilde{x}(s') = e^{\mp s}, \quad \int_0^L ds' K_0(|s-s'|)\bar{x}(s',Z) = Z^{1/2}e^{-Zs}, \tag{3}$$

with $L = N|\ln\alpha_2|$, and K_0 the modified Bessel function of the third kind.

In 1962, Myers[7,8] studied integral equations of type (3) in connection with scattering of electromagnetic waves from a strip. By his method we evaluate (2) explicitly as

$$x_0 = \max(1, \alpha_2) + |\ln\alpha_2|\{\tfrac{1}{8}\theta\eta^{-2}[(1-\eta^2)^2 - (\eta')^2] + \tfrac{1}{4}\eta'\eta^{-1}(1\mp\eta)(1\pm\eta)^{-1} - (1\pm 1)/2\}, \tag{4}$$

where $\theta = L/2$ and $\eta(\theta)$ is a Painlevé[9] function of the third kind which satisfies the nonlinear differential equation

$$(\theta\eta'\eta^{-1})' = \theta(\eta^2 - \eta^{-2}) \tag{5}$$

together with the asymptotic conditions

$$\eta(\theta) \sim -1 + 2\pi^{-1}K_0(2\theta) - 2\pi^{-2}K_0^{\,2}(2\theta) \tag{6a}$$

for large θ, and

$$\eta(\theta) \sim \theta\Omega + \theta^5 2^{-7}(8\Omega^3 - 8\Omega^2 + 4\Omega - 1) \tag{6b}$$

for small θ, where $\Omega(\theta) = \ln(\theta) + \gamma - 2\ln 2$ and γ is Euler's constant. Consider $N' \gg N$; then the identity

$$S_N = \left[\prod_{r=N+1}^{N'} x_0(r)\right] S_{N'} \tag{7}$$

yields the connection away from T_c, where for $S_{N'}$ we use (2.43) and (3.25) of Ref. 3 to obtain

$$S_N^{\pm} \sim N^{-1/4}2^{-1/2}[(1+\alpha_1)/(1-\alpha_1)]^{1/4}\theta^{1/4}[1\pm\eta(\theta)]\exp\{\int_0^\infty dx\, x\ln(x)[1-\eta^2(x)] - g(\theta)\}, \tag{8}$$

where $g(\theta)$ is given by

$$g(\theta) = (2\eta)^{-2}\ln(\theta)\{\theta(\eta^2)' + \theta^2[(1-\eta^2)^2 - (\eta')^2]\}. \tag{9}$$

In order that $S_N^{\pm}$ connect to the $T = T_c$ result, (5.3) of Ref. 3, we take the limit $\theta \to 0$ and find that the following identity must hold:

$$\exp\{\int_0^\infty x\ln(x)[1-\eta^2(x)]\,dx\} = e^{1/4}2^{7/12}A^{-3}, \tag{10}$$

where A is Glaisher's constant, whose integral representation was first derived by Glaisher[10] in 1877 as

$$A^{-3}2^{7/12} = \pi^{1/2}\exp\{-1 - 2\int_0^{1/2}\ln[\Gamma(1+x)]\,dx\}. \tag{11}$$

The behavior of the magnetic susceptibility near the critical temperature is as follows[11,12]:

$$\beta^{-1}\chi = C_{0\pm}|1 - T_c/T|^{-7/4} + C_{1\pm}|1 - T_c/T|^{-3/4} + C_{2\pm} + O(1). \tag{12}$$

Using the "rotational symmetry" discussed above, we sum $\langle\sigma_{00}\sigma_{MN}\rangle$ to obtain *exactly*[13]

$$C_{0+} = D\int_0^\infty \theta\, d\theta\,[1+\eta(\theta)]\exp\{\int_\theta^\infty [1-\eta^2(x)]x\ln(x)\,dx - g(\theta)\}, \tag{13a}$$

$$C_{0-} = D\int_0^\infty \theta\, d\theta\,([1-\eta(\theta)]\exp\{\int_\theta^\infty [1-\eta^2(x)]x\ln(x)\,dx - g(\theta)\}), \tag{13b}$$

where D is given by

$$D = 2^{-1/2}\pi(z_{1c}z_{2c})^{-1}(z_{1c}+z_{2c})^{1/4}\{\beta_c[E_1/(1-z_{2c}) + E_2/(1-z_{1c})]\}^{-7/4}; \tag{14}$$

and

$$C_{1+}/C_{0+} = -C_{1-}/C_{0-} = -R_0\beta_c, \tag{15}$$

with R_0 being given by

$$R_0 = [E_1^{\,2}z_{2c}^{\,2}(1+6z_{1c}^{\,2}+z_{1c}^{\,4}) + E_2^{\,2}z_{1c}^{\,2}(1+6z_{2c}^{\,2}+z_{2c}^{\,4}) - 8E_1E_2z_{1c}z_{2c}(z_{1c}+z_{2c})^2] \times\{8z_{1c}z_{2c}(z_{1c}+z_{2c})[E_1(1-z_{1c}) + E_2(1-z_{2c})]\}^{-1}. \tag{16}$$

In Fig. 1, we plot the dimensionless quantity $R_0(E_1+E_2)^{-1}$ as a function of E_1/E_2. Note that this quantity is $\frac{1}{2}$ at $E_1/E_2 = 0$, and $-\sqrt{2}/16$ at $E_1/E_2 = 1$.

In order to compare with available series results we specialize to the case $E_1 = E_2 = 1$, $z_{1c} = z_{2c}$

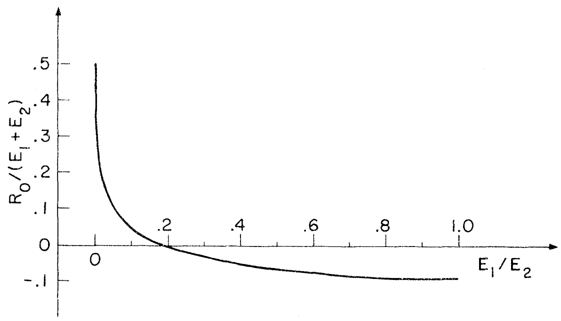

FIG. 1. The dimensionless quantity $R_0/(E_1+E_2)$ as a function of $x=E_1/E_2$. When $x=1$, the value is $-\sqrt{2}/16$. As $x\to 0$ the function behaves as $1/2+3/2\ln x$. The curve crosses zero at ~ 0.1993.

$=\sqrt{2}-1$, $\beta_c=\frac{1}{2}\ln(1+\sqrt{2})$, and numerically evaluate (13)–(15) to obtain[14]

$$C_{0-}=0.025\,536\,971\,9\ldots,$$

$$C_{0+}=0.962\,581\,732\,2\ldots,$$

$$C_{1-}=-0.001\,989\,410\,7\ldots,$$

$$C_{1+}=0.074\,988\,153\,8\ldots.$$

These results give very good agreement with Sykes *et al.*[15] above T_c, $C_{0+}=0.962\,59\pm 3\times 10^{-5}$, $C_{1+}=0.0742$, and with Guttmann[16] below T_c, $C_{0-}=0.0256\pm 1\times 10^{-4}$.

Arguments can be made that the constants C_{2+} and C_{2-} are equal. The value of this constant depends on correlation functions at short distances, and hence cannot be computed by the present method. Details will be published elsewhere.

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Wilson Theory for Spin Systems on a Triangular Lattice

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A special renormalization transformation is constructed for one-component spin systems on a two-dimensional triangular lattice. Fixed point, eigenvalues, and eigenvectors are determined in various approximations, which converge well to known Ising data.

Most of the specific results of the renormalization approach to critical phenomena have been obtained by the ϵ expansion for continuous spin systems interacting through a Landau-Ginzburg Hamiltonian[1] (with $\epsilon=4-d$, and d the dimensionality of the spin lattice). This Letter concerns an application of Wilson's[2] ideas to a general class of discrete spin Hamiltonians which comes closer to Kadanoff's[1] original derivation of the scaling laws and which avoids the ϵ expansion (which is presumably asymptotic rather than convergent).

The method is best illustrated[3] for a two-dimensional (2D) triangular lattice. In Fig. 1 the lattice is divided into cells (triangles) having an *odd*