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[9] Additional perturbation terms are suppressed by ordering factors Λ_i/L, where Λ_i refers to the particle excursion lengths per wave cycle. In the Lorentz frame for which the longitudinal and transverse particle velocity vanish simultaneously, $\Lambda_\perp = \lambda\mu$ and $\Lambda_\parallel = \lambda\mu^2$, where $\mu \equiv (e/mc)A_\perp$.

[10] A comparable result has been obtained by T. W. B. Kibble [Phys. Rev. 150, 1060 (1966)]. His analysis treats the average motion of a charged particle in an electromagnetic field in a weak-field limit [$(e/m)A_\mu$ is an expansion parameter]. The present perturbation scheme treats arbitrarily strong electromagnetic fields; our perturbation parameters are the quantities Λ_i/L, as described above.

Observation of Temperature Oscillations in Quasi-isothermal Superfluid Film Oscillations*

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We report the first observations of the small temperature oscillations which accompany the *nearly* isothermal oscillations in the levels of two reservoirs of He II which are connected only by the mobile superfluid film. We show that our experimental results are in accord with a nearly isothermal approximation to the general theory first elucidated by Robinson for capillary flow. The amplitude of the temperature oscillations is in agreement with expectations based on the measured thermal conductance and time constant for our apparatus.

Consider two reservoirs each partially filled with He II. Let the reservoirs be coupled only by means of the mobile superfluid film. As is well known, any chemical potential difference between the fluid surfaces in the reservoirs will tend to be reduced by mass transport through the superfluid film. As explained by Robinson[1] in his treatment of He II flow in narrow channels, the kinetic energy of the moving superfluid may result in an oscillation in the level difference between the two baths. In particular, Robinson showed that the two cases of weak and strong thermal contact between the reservoirs resulted in oscillatory solutions to the equations of motion. He called these two cases adiabatic and isothermal. Atkins[2] was the first to observe the isothermal oscillations in a film-flow experiment and the adiabatic oscillations were first seen in bulk flow through superleaks by Manchester and Brown.[3] More recently, Hammel, Keller, and Sherman,[4] Hoffer *et al.*,[5] and Hallock and Flint[6] have studied various aspects of the "isothermal" film oscillations. Here we wish to report the first observation of the small-amplitude temperature oscillations which accompany the level oscillations described by the nearly isothermal limit of the theory due to Robinson.[1]

The apparatus, Fig. 1, consists of two cylindrical reserviors each of which makes up the annular region of a coaxial capacitor. The reservoirs are sealed except for a superfluid film path. A capillary (not shown) which has a cross sectional area which is 6% of the area of the annular region enters from the bottom and is used to fill the assembly to about the midpoint of the annular region. A superfluid valve[7] is used to seal the

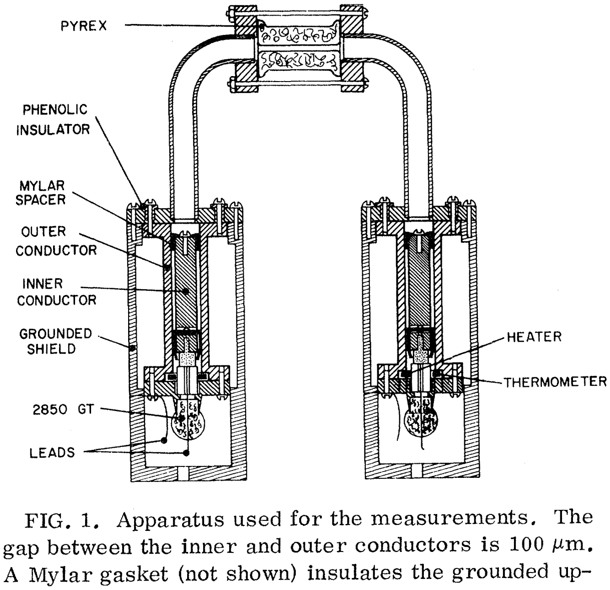

FIG. 1. Apparatus used for the measurements. The gap between the inner and outer conductors is 100 μm. A Mylar gasket (not shown) insulates the grounded upper arm from the capacitors. Superfluid integrity is accomplished through the use of indium seals.

capillary during measurements. Electrical leads for the heater, thermometer, and inner conductor are removed through feedthroughs[8] which employ Stycast[9] 2850GT. Flow along the superfluid film path is limited by the presence of a Pyrex channel having an inside diameter of 0.040 in. and a length of $\frac{1}{2}$ in. The He II level in either reservoir can be measured by means of a three-terminal ac capacitance technique which employs a General Radio 1615A capacitance bridge (operating at 1 kHz, 15 V rms) and PAR HR-8 as a null detector. Level measurements in a given reservoir can be made with a sensitivity of ±200 Å. Temperatures are measured *within* each reservoir and temperature differences are measured with a fully shielded differential Wheatstone bridge of standard design. The temperature sensors are $\frac{1}{10}$-W Allen-Bradley resistors.

The entire apparatus of Fig. 1 is contained within a can which can be sealed, and since the helium within the reservoirs itself constitutes a sealed system, the sealed can is filled with superfluid to above the Pyrex channel. This helps to ensure a nearly isothermal environment for the reservoirs and film flow path. The sealed can was located at the end of a standard Dewar probe and the whole in a pumped ^{4}He Dewar. During differential temperature measurements the ambient temperature in the sealed can was measured to be stable to ± 0.3 μK/h. This stability was accomplished by using a carefully constructed and thermally shielded Walker regulator and electronic feedback stabilization in the main Dewar bath. To avoid further the effects of drifts in the ambient temperature the resistors used as sensors for the differential measurements were carefully selected[10] so as to have nearly identical temperature coefficients.

An example of the oscillation in temperature difference between the two reservoirs and the oscillation in level which produces it is shown in Fig. 2. To produce this trace a dc voltage applied to one of the center conductors at an earlier time was removed. The steady dc voltage had resulted in a *level* difference between the two baths but no chemical potential difference. Removal of the voltage results in a nearly instantaneous chemical potential difference given by the level difference. The resulting oscillation in level gives rise to the temperature difference oscillation.

Least-squares computer fits to the two oscillations shown in Fig. 2 result in $\omega_T = 0.1604 \pm 0.0052$ rad/sec and $\omega_L = 0.1629 \pm 0.0002$ rad/sec for the temperature and level oscillations, respectively. These are seen to be equal within experimental error. The two oscillations differ in phase by $\Delta\varphi \approx \pi/2$ which agrees with expectations. Note that since the pens were offset on our recorder the traces *look* out of phase by π but this is not the case. As was the case in our earlier observations[6] involving apertures and short channels, we do not observe any *strong* shifts[4] in the frequency and decay constant of the level oscillations as these oscillations diminish. We do, however, generally see *gradual* (a few percent) shifts in the frequency and decay constant as a given oscillation progresses. These shifts are outside the scope of the present report and will be discussed in a future publication.

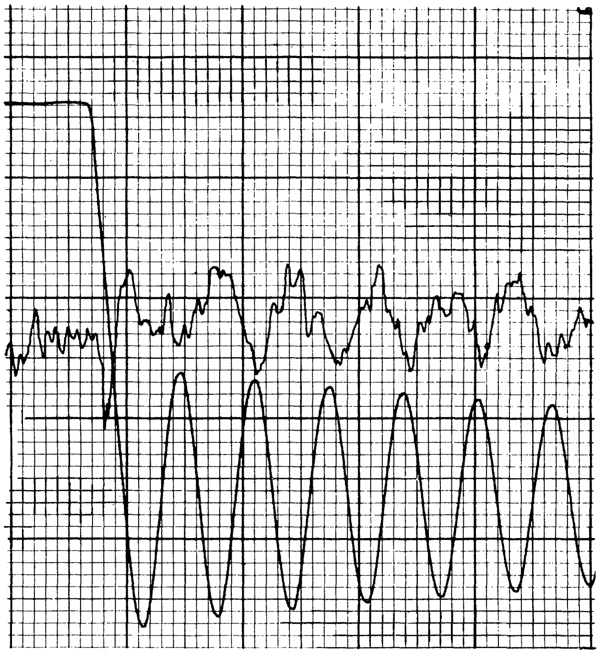

FIG. 2. Temperature and level oscillations at 1.239°K. Upper trace, differential temperature between the two reservoirs. Temperature changes within a single reservoir are thus half those shown. Lower trace, displacement x of the helium level in one reservoir relative to the equilibrium position. For time, temperature, and level one small square represents 6 sec, 0.17 μ°K, and 10 μm, respectively. Note that the chart pens are offset by about $1\frac{1}{2}$ small squares and hence the oscillations only appear to be out of phase by π and are actually $\pi/2$ out of phase.

That these oscillations are of the isothermal class described by Robinson[1] is confirmed by a comparison of the amplitude of the observed temperature oscillations expectations based on measurements of the thermal conductance K and thermal time constant τ for our reservoirs and by a study of the temperature dependence of the decay constant and frequency of the level oscillations. A linearized, nearly isothermal approximation to Robinson's theory, when applied to our geometry, results in an equation for the level oscillations of the form

$$\ddot{x} + 2\alpha\dot{x} + \omega^2 x = 0,$$

where

$$2\alpha = \frac{\omega^2\rho V C_p}{K}\left\{1 + \frac{TS^2 a}{gVC_p}\left[1 - \frac{1}{\rho S}\left(\frac{dp}{dT}\right)_{\text{sat}}\right]\right\},$$
$$\omega^2 = (4\pi\rho_s g/\rho a)\left[\int ds/r(s)\delta(s)\right]^{-1}. \qquad (1)$$

Here x is the displacement of the helium level in one reservoir relative to the equilibrium position, ρ the density,[11] T the temperature, S the entropy,[12] g the gravitational acceleration, and ρ_s the superfluid density;[13] $\delta(s)$ is the position-(s) dependent superfluid film thickness,[14] C_p the specific heat,[12] V the volume of He II in our reservoir, $(dp/dT)_{\text{sat}}$ the temperature derivative of the vapor pressure curve,[15] and $r(s)$ the radius of the channel through which the film flows. Factors which relate directly to our detectors are K and a, the cross-sectional area of the annular region. The integral is along the path taken by the film.

Aside from a possible weak dependence of the film thickness on temperature, the decay constant and frequency have a temperature dependence given by the known quantities ρ_s/ρ, C_p, $(dp/dT)_{\text{sat}}$, and S and quantities we measure, K and T. We have made careful measurements of the temperature dependence of K for our apparatus and find $K \sim T^{4.2}$. Thus, the temperature dependence of the decay constant and frequency can be predicted. The dominant contribution to the decay constant ($\approx 90\%$) comes from the term $\omega^2\rho TS^2 a/gK$ in Eq. (1). The agreement between our observations and the isothermal limit of the Robinson theory can be seen by reference to Fig. 3. We should note here that to obtain the data represented by the triangles we have carried out least-squares computer fits with the functional form $x = A\exp(-\alpha t)\sin(\Omega t + B)$. The undamped frequency ω is then obtained from α and Ω in the usual manner.

Had these oscillations been of the adiabatic class, the temperature dependence of their frequency would have been quite different. Robinson has shown that the adiabatic frequency ω_A is related to the isothermal frequency ω by the expression $\omega_A = \omega(1+\gamma)^{1/2}$, where

$$\gamma = \frac{TS^2 a}{gVC_p}\left[1 - \frac{1}{\rho S}\left(\frac{dp}{dT}\right)_{\text{sat}}\right]. \qquad (2)$$

For the parameters of our experiment the ω_A values differ strongly from the observed ω values in the temperature range investigated. This can be seen by reference to the inset in Fig. 3(b). Note that the ordinate of the inset starts at zero while that for the larger part of Fig. 3(b) does not. The temperature axis covers the same range in each case. The agreement in temperature dependence between theory and experiment provides us with convincing evidence that the observed temperature oscillations are associated with isothermal or, more correctly, quasi-isothermal and not adiabatic conditions.

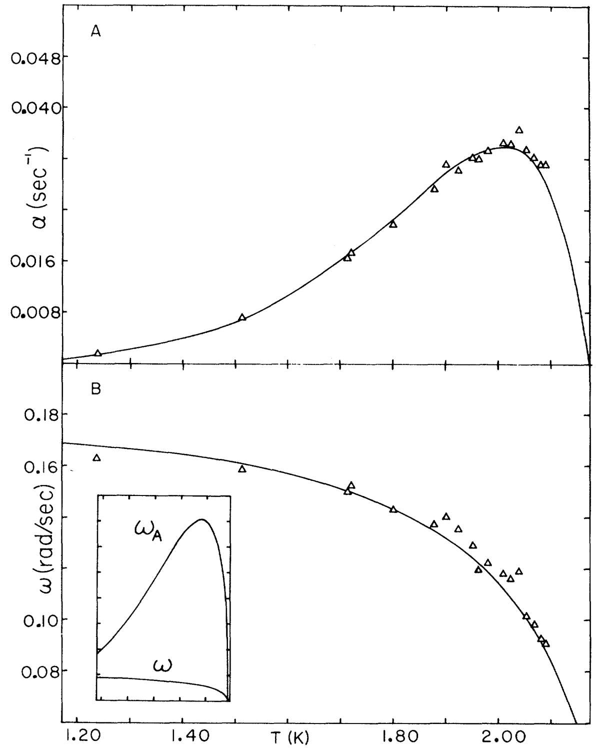

FIG. 3. (a) Decay constant versus temperature; (b) frequency versus temperature. The points represent the average of several measurements at each temperature. Solid curves, calculated from the nearly isothermal approximation to the theory due to Robinson. Note that the lower limit of the ω axis is 0.06 rad/sec and not 0. The inset in (b) compares the isothermal frequency ω with the frequency ω_A which would have been observed had we carried out our measurement under adiabatic conditions.

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Bäcklund Transformation for Solutions of the Korteweg-de Vries Equation*

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We present a Bäcklund transformation operating on solutions of the Korteweg–de Vries equation. It provides a unifying approach to the theory of solitary waves (solutions). Recursive application of the transformation to *any* solution of the Korteweg-de Vries equation generates a hierarchy of solutions for which an algebraic recursion relation is obtained. Every such hierarchy (one of which is the known family of pure multisoliton solutions) thus admits a nonlinear superposition principle.

Recent analyses have shown that the first-order effects of dispersion and dissipation in a vast range of nonlinear physical systems can be described by a few simple nonlinear equations of evolution, such as the Burgers, sine-Gordon, and Korteweg–de Vries (KdV) equations,[1] and in each case these equations have been found to possess stable steady-state solutions with remarkable properties. Considerable effort is going into an attempt to understand the mathematical nature of these elementary excitations, including their surprising superposability and stability throughout interactions.[1-4]

A uniform approach to these problems, which can be applied systematically to all such equations, has not really been achieved as yet, but it appears that the transformation and invariance properties of the equations may be the key principle. The Burgers equation, for example, is directly transformable to the linear diffusion equation.[5] The sine-Gordon equation, which itself apparently first arose in connection with a transformation problem in differential geometry,[6] has long been known to admit a Bäcklund transformation from which many of its interesting properties can be derived (Ref. 6, and Lamb, Ref. 1). The discovery of an equally simple Bäcklund transformation for the third-order KdV equation strengthens our conviction that it is from transformation theory that a fundamental understanding of these equations will result.[7]

Solitons.—The Korteweg–de Vries equation