within the experimental uncertainty. The final rise in the curve represents the zero-sound regime; in this regine, however, Eq. (3) ceases to be valid. We also would like to comment on the excellent fit that was obtained above the knee, which justifies to some degree our choice of v_{is}. It should be realized that a more accurate theory would have to take into account the hexagonal symmetry of the crystal as well as the quantum solid character of solid helium. These calculations are at present not available.

The one outstanding difficulty in comparing Niklasson's theory with experiment is the large ratio τ^1/τ^{11} required. To some degree this may reflect the use of an isotropic version of the theory where proper hexagonal symmetry should have been used. We feel, in spite of this, that the effect is too large to be exclusively due to this source. We want to remark here that τ^1 corresponds to a length $l = v_2\tau^1$, where v_2 is the second-sound velocity, giving the range of the coupling between local phonon flow and lattice deformations. Our results seem to indicate that this range is much larger in solid helium than in a quasiharmonic solid with anharmonic corrections for which the theory of Niklasson was developed.

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Exact Solution of an Ising Model with Three-Spin Interactions on a Triangular Lattice

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The Ising model on a triangular lattice with three-spin interactions is solved exactly. The solution, which is obtained by solving an equivalent coloring problem using the Bethe *Ansatz* method, is given in terms of a simple algebraic relation. The specific heat is found to diverge with indices $\alpha = \alpha' = \frac{2}{3}$.

An outstanding open problem in lattice statistics has been the investigation of phase transitions in Ising systems which do not possess the up-down spin-reversal symmetry.[1,2] A well-known example which remains unsolved to this date is the Ising antiferromagnet in an external field. Another problem of similar nature that has been considered recently[3-5] is the Ising model on a triangular lattice with three-body interactions. This latter model is self-dual so that its transition temperature can be conjectured[3,6] using the Kramers-Wannier argument.[7] However, the nature of the phase transition has hitherto not been known.

We have succeeded in solving this model exactly. In this paper we report on our findings. It will be seen that the results are fundamentally different from those of the nearest-neighbor Ising models. While the final expression of our solution is quite simple, the analysis is rather lengthy and involved. For continuity in reading, therefore, we shall first state the result. An outline of the steps leading to the solution will also be given.

Consider a system of N spins $\sigma_i = \pm 1$ located at the vertices of a triangular lattice L. The three spins surrounding every face interact with a three-body interaction of strength $-J$, so that the Hamiltonian reads

$$\mathcal{H} = -J\sum \sigma_i\sigma_j\sigma_k, \tag{1}$$

with the summation extending over all faces of L. Let Z be the partition function defined by (1). We find the following expression for $Z^{1/N}$ in the

thermodynamic limit:

$$W = \lim_{N\to\infty} Z^{1/N} = (6y_t)^{1/2}, \tag{2}$$

where $t = \sinh 2K$, with $K = |J|/kT$, and $1 \leq y < \infty$ is the solution ot the algebraic equation

$$(y-1)^3(1+3y)/y^3 = 2(1-t)^4/t(1+t^2). \tag{3}$$

The function $W(t)$ is unique since y is single valued in t for $0 \leq t < \infty$. It is also seen that $\ln W$ has a singular part which behaves as $|t-1|^{4/3}$ near $t=1$. It follows that a phase transition occurs at $t=1$ or $kT_c/|J| = 2/\ln(\sqrt{2}+1) = 2.269\,185\ldots$. This confirms the duality prediction[3,6] which merely reflects the invariance of the right-hand side of (3) under the transformation $t \to t^{-1}$. The energy and specific heat per spin, E and c, respectively, are plotted in Fig. 1. The energy has the critical value $E_c = -\sqrt{2}|J|$, while the specific heat diverges near T_c as

$$c \sim \tfrac{2}{9}k[\ln(\sqrt{2}+1)]^{4/3}|1 - T/T_c|^{-2/3}. \tag{4}$$

This is in contradistinction to the logarithmic divergence of the nearest-neighbor models, and to the result $\alpha = \alpha' = \frac{1}{2}$ of the isotropic three-spin model on the "Union Jack" lattice.[8]

Two key steps are involved in our approach. The Ising model (1) is first converted into a coloring problem; the latter is subsequently solved using a generalization of the Bethe *Ansatz* method.[9] We now outline these two steps separately.

Conversion into a coloring problem.—The triangular lattice L is composed of three sublattices L_1, L_2, and L_3 with the property that the sites of L_i and L_j form a hexagonal lattice L_{ij}. Two such lattices, L_{13} (solid lines) and L_{23} (broken lines), are shown in Fig. 2.

We first carry out a dual transformation for the spins on L_{23} while leaving the spins on L_1 unchanged. The effect of the dual transformation is to introduce an additional variable $\mu = \pm 1$ to each face of L_{23}.[10] Thus we may specify the states of the sites of L_1 by the four-valued variable (σ, μ). Furthermore, to each neighboring pair (σ_i, μ_i) and (σ_j, μ_j) of L_1, the dual transformation yields a weight factor

$$\omega_{ij} = 2^{-1/3}\{\exp[K(\sigma_i+\sigma_j)] + \mu_i\mu_j \exp[-K(\sigma_i+\sigma_j)]\}. \tag{5}$$

The partition function is then given by

$$Z = \sum\prod\omega_{ij}, \tag{6}$$

where the summation is extended over all states (σ_i, μ_i) and the product over all nearest neighbors of L_1.

Next, for each nonvanishing term in (6) we associate colors 1, 3, 5, 7 to the sites of L_1 according to the rule

$$\begin{aligned}&(+,+) = \text{color } 1, \quad (-,+) = \text{color } 3,\\ &(-,-) = \text{color } 5, \quad (+,-) = \text{color } 7.\end{aligned} \tag{7}$$

It is clear from (5) that two neighboring sites of L_1 cannot be colored 1, 5 or 3, 7. Thus we may further color the sites of L_3 with colors 2, 4, 6, 8 under the restriction that the colors of neighboring sites on L_{13} differ by exactly 1 (to modulus 8, i.e., 8 and 1 can be adjacent). Note that we have just completed a site coloring for L_{13}. Now we introduce activities to the colors:

$$\begin{aligned}&z_1 = z_3 = z_5 = z_7 = 1\\ &z_2 = 1/z_4 = z_6 = 1/z_8 = \sinh 2K.\end{aligned} \tag{8}$$

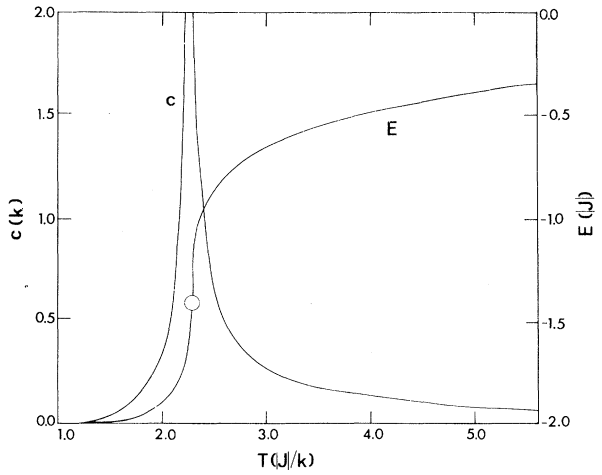

FIG. 1. Energy (E) and specific heat (c) per spin.

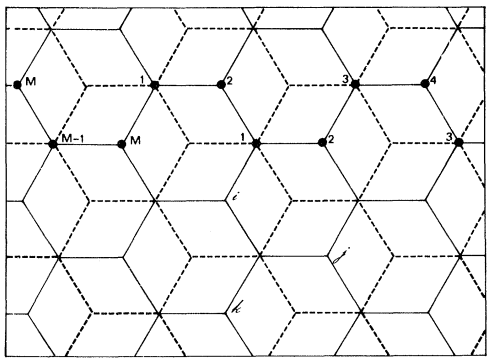

FIG. 2. Decomposition of L. Solid lines denote L_{13}, broken lines denote L_{23}.

It can be verified that the activities of the possible colorings of a site on L_3 surrounded by sites (σ_i, μ_i), (σ_j, μ_j), and (σ_k, μ_k) of L_1 (cf. Fig. 2) generate precisely the product $(2\sinh 4K)^{-1}\omega_{ij}\omega_{jk}\omega_{ki}$. Using (6), we are thus led to the identity

$$Z = (2\sinh 4K)^{N/3} Z_c, \tag{9}$$

where

$$Z_c = \sum z_1^{n_1} z_2^{n_2} \cdots z_8^{n_8} \tag{10}$$

is the generating function for the site coloring of L_{13}.

Solution of the coloring problem.—Draw arrows on the edges of the dual triangular lattice L_D of L_{13}, pointing to an observer's left (right) if the colors on L_{13} increase (decrease) as he crosses the arrow. Then at each vertex of L_D there are three arrows in and three out. Hence our coloring problem bears the same relation to the "triangular ice" model[11] as the three colorings of the square lattice[12] do to the "square ice"[13] model. In particular, since $z_1 = \ldots = z_8 = 1$ when $T = T_c$ and $t = 1$, it follows that at the critical temperature the coloring and triangular ice models are equivalent, so from (9) and Eq. (41) of Ref. 11 one obtains $W_c = \sqrt{6}$. This agrees with (2).

To obtain W for the more general activities (8), number the sites in each row of L_{13} as in Fig. 2, using cyclic boundary conditions as indicated. Let $C = \{c_1, \ldots, c_M\}$ be the coloring of the upper row of sites $1, \ldots, M$, and $C' = \{c_1', \ldots, c_M'\}$ the coloring of the lower row. Introduce the transfer matrix

$$A(C, C') = \prod_{i=1}^{M} (z_{c_i} z_{c_i'})^{1/2} \tag{11}$$

if the colors of adjacent sites differ by 1; otherwise $A(C, C') = 0$.

Consider a basic sequence $\{m, m+1, m+4, m+5, m+8, m+9, \ldots\}$ of colors in a row, which corresponds to a row of arrows pointing up on L_D, and then introduce n dislocations at positions $X = \{x_1, \ldots, x_n\}$ in the sequence so as to define C by the rules

$$c_1 = m, \quad c_{x+1} = c_x + 2 + (-1)^x - 2n_x, \tag{12}$$

where $x = 1, \ldots, M$ and n_x is the number of dislocations in position x (between sites x and $x+1$). To be consistent with the coloring rule, X must lie in the domain D:

$$1 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq M, \tag{13}$$

no two odd x_j's equal, no four even x_j's equal.

These dislocations correspond to there being arrows pointing down on the efges of L_D. There are the same number n in every row, so we can consider some fixed value of n between 0 and $2M$.

Let Λ and f be an eigenvalue and eigenvector of A, respectively, and $f(m, X)$ the element of f corresponding to the row coloring (12). Then the eigenvalue equation is

$$\Lambda f(m, X) = \sum_Y \left(\prod_{j=1}^{n} w(m + x_j + y_j - 2j) \right) \times f(m+2, Y), \tag{14}$$

where

$$w(m) = (z_m z_{m+1})^{1/2}, \tag{15}$$

$$f(m+2, 0, x_2, \ldots, x_n) = f(m, x_2, \ldots, x_n, M), \tag{16}$$

and the summation is over all $Y \in D$ such that

$$|y_j - x_j + 1| \leq \tfrac{1}{2}[1 + (-1)^{x_j}], \tag{17}$$

and if $x_j = x_{j+1}$, then $y_j \neq y_{j+1}$. For a large lattice of $2N/3$ sites,

$$Z_c^{3/2N} = \Lambda_{\max}^{1/M}, \tag{18}$$

where $\Lambda_{\max}$ is the maximum eigenvalue of (14). We try the generalized Bethe *Ansatz*

$$f(m, X) = \sum_P a(P) \prod_{j=1}^{n} \varphi_{P_j}(m - 2j, x_j), \tag{19}$$

where the summation is over all $n!$ permutations $P = \{P_1, \ldots, P_n\}$ of the integers $1, \ldots, n$. The coefficients $a(P)$ and the n functions $\varphi_j(m, x)$ are at our disposal. We require that there exist n wave numbers $k_1, \ldots, k_n$ such that

$$\varphi_j(m, x) = \varphi_j(m+4, x) = \exp(2ik_j)\varphi_j(m, x-2) \tag{20}$$

for all integers m, x. We find that the *Ansatz* works: Let E_j be defined by

$$\cosh 2E_j = \cos 2k_j + t + t^{-1}; \tag{21}$$

then Λ and $k_1, \ldots, k_n$ are given by

$$\ln\Lambda = \sum_{j=1}^{n} (E_j - ik_j), \tag{22}$$

$$\exp(iMk_j) = -\prod_{l=1}^{n} B_{jl}, \quad j = 1, 2, \ldots, n, \tag{23}$$

with $B_{jl} = -\cosh(E_j + ik_l)/\cosh(E_l + ik_j)$. If $k_1, \ldots, k_n$ are real, then from (21) B_{jl} is unimodular. Also, by using elliptic functions we can change variables from k_j to u_j so as to make B_{jl} a function only of $u_j - u_l$.

For $\Lambda_{\max}$ the equations (23) can therefore be

solved in the limit of M, n large by methods similar to those used for the normal Bethe *Ansatz*.[13,14] Using (22), (18), and (9) we then obtain the result (2).

We intend to give details elsewhere. We remark that $k_1, \ldots, k_n$ are real and distributed symmetrically over the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, that $n = M$, and that $E_1, \ldots, E_n$ are real and non-negative.

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Added note.—Numerical estimates of the critical indices α', β, and γ' of the present model have recently been given by H. P. Griffiths and D. W. Wood, J. Phys. C: Proc. Phys. Soc., London 6, 2533 (1973).

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Clarification of Local-Moment–Conduction-Electron Resonance*

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Our data on spin resonances in *Ag*-Er are inconsistent with the $J\vec{S}\cdot\vec{s}$ model, and barely consistent with $J(\hat{k}, \hat{k}')$. Alternatively, a suggestive temperature dependence of the $J\vec{S}\cdot\vec{s}$ conduction-electron g value is compatible with the data. We present general kinetic equations for a local-moment–conduction-electron system. We clarify questions of detailed balance, relaxation to exchange fields, positive energy absorption, and reduction of the kinetic equations for any possible model to either of two standard forms.

We present results for the electron spin-resonance parameters for the local-moment–conduction-electron spin system *Ag*-Er. This system is characterized by two widely separated resonances (the g value for the conduction electrons is $\simeq 2.0$, while for Er, $g \simeq 6.8$), which are coupled via an exchange interaction. In the course of our efforts to analyze this system we have completed a theoretical and computational study which we believe resolves important questions, and raises new ones, concerning the dynamical description of a dilute local moment in a metal.

The experiments were performed utilizing the transmission electron spin-resonance (TESR) technique.[1] The samples were typically single-crystal foils 0.002 cm thick containing between 20 and 70 ppm of Er in Ag.[2] The sample temperature was varied between 1.25–30°K. The signal in the TESR technique is defined as the projection of the transmitted microwave field on a reference field of the same frequency. The data are taken at fixed temperature, frequency, and reference phase angle, while the applied dc field is slowly swept through the resonance conditions.

In Fig. 1 we present such a transmitted signal versus dc field where the phase of the reference has been chosen so as to yield a symmetric high-field resonance (HFR). The low-field resonance (LFR) is clearly seen and for the particular set of conditions also appears symmetric.[3] The general properties of the resonances are the following: The LFR amplitude ($g \cong 6.84$) is slightly larger than the HFR at 1.3°K. The LFR broadens linearly with temperature, and the amplitude de-