$+(z^4-1)^{1/2}]^{1/3}$, $L \equiv \frac{1}{2}[(U+4/U)-4z(U+4/U)^{-1/2}]$, $B \equiv 2/[(U+4/U)^{1/2}-(U+4/U-4L)^{1/2}]$, $B_x = \mathrm{Re}(B)$, $B_y = -\mathrm{Im}(B)$, $A = \mathrm{Im}[3(B^2+B^{-2})/8]$.

[9]D. J. Rose and M. Clark, Jr., *Plasmas and Controlled Fusion* (Massachusetts Institute of Technology Press, Cambridge, Mass., 1961).

Anomaly in the Longitudinal Sound Velocity of hcp He^4

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Measurements of the longitudinal sound velocity of single crystals of hcp He^4 are reported from 0.7 K to the melting temperature at melting pressures of 86 and 119 bar. The temperature dependence changes from a power law to approximate temperature independence below about 1.1 K at $p_m = 86$ bar and below about 2 K at $p_m = 119$ bar. It is believed that this marks the transition from adiabatic sound propagation to a region where coupling to the temperature field is important.

We are reporting on measurements of the sound velocity of single crystals of hcp He^4 from 0.75 K to the melting temperature at densities corresponding to melting pressures of 86 and 119 bar. In most of the crystals studied, the temperature dependence of the sound velocity shows an abrupt change at some point between 1 and 2 K.

Crystals of hcp He^4 were grown at constant pressure over a period of 24 h. After initial growth, the crystals were annealed in a temperature gradient of about 1 K. This annealing procedure frequently produced crystals of very high quality. Near the melting temperature we have observed up to 160 ultrasonic echoes in these crystals. We believe that many of the crystals were single. The crystals were contained inside a beryllium-copper pressure cell, which also contained the ultrasonic etalon. Measurements were performed at 5 MHz provided by a longitudinally cut quartz tranducer in its fundamental. A pulse reflection method was used, with a transducer to mirror distance of 0.957 cm. This distance was obtained by calibrating against the sound velocity in liquid He^4 using the results of Vignos and Fairbank.[1] The reflecting mirror was the bottom of the pressure cell, polished flat to 2 wavelengths of sodium light.

An absolute value for the sound velocity was obtained near the melting point from a least-squares fit to the arrival times of about twenty echoes. We estimate the error in this measurement at about 2%. The change in sound velocity with temperature was measured by observing the phase shift in the high-frequency signal to 1 part in 10^4. For this purpose the high-frequency signal was displayed on an oscilloscope with the delayed time base triggered coherently with the pulsed oscillator providing the ultrasonic signal.

Temperature measurements were made with a Solitron germanium thermometer calibrated against the vapor pressures of He^3 and He^4.

The sound velocity increases with falling temperature, as seen in Figs. 1 and 2. In most crys-

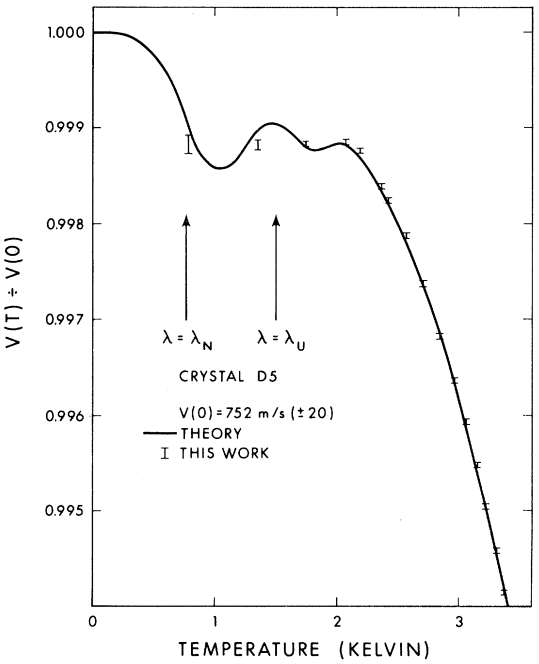

FIG. 1. $v(T)/v(0)$ for a crystal grown at 119.6 bar. Solid line, least-squares fit by Eq. (3) with $\tau^{-1} = 23\tau^{-11}$, $\tau_U = 1.5\tau_U(\mathrm{expt})$, and $\tau_N = 1.5\tau_N(\mathrm{expt})$.

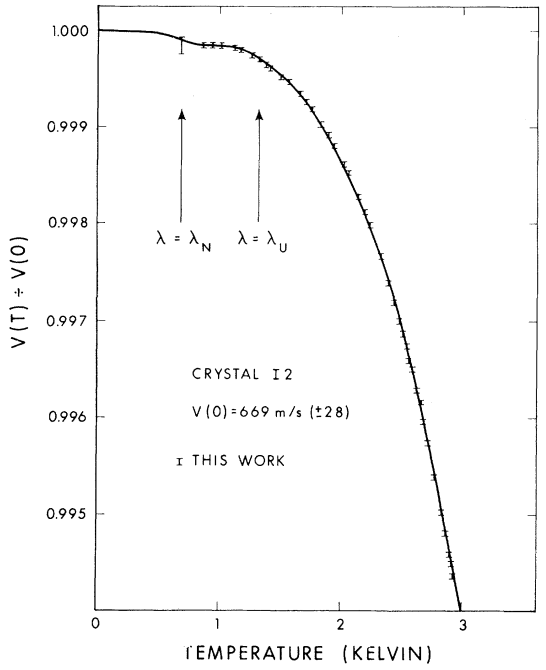

FIG. 2. $v(T)/v(0)$ for a crystal grown at 86.4 bar. Solid line, least-squares fit by Eq. (3) with $\tau^{1}=1.9\tau^{11}$, $\tau_U=0.2\tau_U(\text{expt})$, and $\tau_N=1.5\tau_N(\text{expt})$.

tals we observe a rather abrupt flattening out of the velocity as a function of temperature. We refer to the position where this occurs as the "knee." Possible small variations of the velocity in this range could not be detected with our technique because of the drastically increased attenuation that is observed in the region of the knee and below. In Table I we show the position of the knee for all cases observed. Included in the table is the absolute velocity at 0 K [obtained through extrapolation using Eq. (9) below]. We compared these velocities with velocities calculated from elastic constants[2] in order to obtain an estimate of the angle between the hexagonal axis and the direction of sound propagation. For all crystals grown at 119 bar this angle lies in the range 40° to 90°. In this range the velocity is double valued and changes relatively slowly with orientation. In view of this we are not able to establish the orientation of these crystals beyond the above estimate. For the measurements at 86 bar the crystal showing the fast velocity (778 m/sec) appears to be close to 0°, whereas the other two can be anywhere in the range 40° to 90°. Because of these large uncertainties we are at present not able to draw any conclusions about the influence of orientation on the observed effect.

TABLE I. Knee temperature.

Pressure (bar)	Absolute velocity (m/sec)	Knee position (K)
119.5	736 ±2%	2.5 ±0.1
119.2	755	2.8
119.6	745	2.1
119.6	751	2.3
117.8	757	1.9
119.0	717	1.8
119.2	747	2.1
86.2	778	1.2
86.6	680	1.1
86.4	669	1.1

The characteristics of sound propagation in a dielectric depend on interactions between the sound wave and the phonon bath. We characterize this interaction by two phonon mean free paths: λ_R for resistive (i.e., not momentum conserving) processes, and λ_N for normal processes. For adiabatic sound propagation the sound wavelength λ must be in the range

$$\lambda \gtrsim \lambda_R. \tag{1}$$

λ_R increases with falling temperature, and sound will therefore not propagate adiabatically below a temperature given by $\lambda \approx \lambda_R$.[3] If at this temperature one also has $\lambda \lesssim \lambda_N$, the propagation will become collisionless. It is, however, also possible, depending on the wavelength λ, that a so-called "second sound window" exists:

$$\lambda_N < \lambda < \lambda_R. \tag{2}$$

In the temperature range in which the condition (2) holds, second sound can be excited *in situ* (the condition for its observation over macroscopic distances is more stringent).[4] In this case the transition from adiabatic to collisionless sound will take place over an extended temperature range. In the transition region coupling to the temperature field is important.

The phonon mean free paths λ_R and λ_N of hcp He^4 have been experimentally determined from thermal conductivity[5,6] and second-sound data.[4] For λ_N we used the data of Hogan, Guyer, and Fairbank.[5] λ_R was assumed to be essentially equal to the umklapp mean free path λ_U, for which we used at 86 bar the extensive data of Lawson and Fairbank.[6] To obtain data for λ_U at 119 bar we extrapolated the data of Ref. 6 according to the suggestions of Hogan, Guyer, and Fairbank. The procedure probably introduces some

error into the λ_U at this pressure. λ_U is anisotropic; we have here used the average for a powder. In Figs. 1 and 2 we indicate the temperatures at which $\lambda = \lambda_N$ and $\lambda = \lambda_U$. It can be seen that a second-sound window does exist at both densities and that it covers approximately the range of the anomalous velocity behavior. Because of the anisotropy of λ_U (as well as other effects like remaining lattice imperfections), one expects certain scatter in the onset of the transition region. This scatter is observed, but at present we cannot make any definite comment on its dependence on the propagation direction. We conclude from this comparison that the observed anomaly is the transition from adiabatic sound propagation to a regime where coupling to the temperature field is important. The measurements did not extend to low enough temperatures to observe the transition to collisionless sound.

In order to discuss this in more detail we make use of the work of Niklasson.[7] For the isotropic case one can obtain from Niklasson's results the following expression for the sound velocity:

$$\left(\frac{v}{v_{is}}\right)^2 = 1 + \frac{1/\rho\kappa_{is}}{v_{is}^2}\left(\frac{C_p}{C_v} - 1\right) \times \frac{(1-s^2)\omega^2 + 4\Gamma^0\Gamma_2{}^0}{(1-s^2)^2\omega^2 + 4(\Gamma_2{}^0)^2}. \quad (3)$$

Here ρ is the density, κ_{is} is the isothermal compressibility, s is the ratio of second-sound to first-sound velocity and is given in terms of the longitudinal and transverse velocities by

$$s = \frac{v_2}{v} = \left(\frac{1}{3}\frac{1+2(v_l/v_t)^3}{1+2(v_l/v_t)^5}\right)^{1/2}. \quad (4)$$

The damping constants Γ^0 and $\Gamma_2{}^0$ are given by

$$\Gamma^0 = \frac{1}{2}\left(\frac{1}{\tau_R} + s^2\omega^2\tau^{11} - 2s^2\omega^2\tau^1\right), \quad (5a)$$

$$\Gamma_2{}^0 = \frac{1}{2}\left(\frac{1}{\tau_R} + s^2\omega^2\frac{\tau^{00}+\tau^{11}}{1+\tau^{00}/\tau_R}\right). \quad (5b)$$

The various relaxation times in these expressions represent different processes and are given by

$$1/\tau_R = 1/\tau_U + v/d, \quad (6)$$

where τ_U is the umklapp relaxation time, d is of the order of the cell dimension, and the term v/d represents boundary scattering;

$$\tau^{00} = 2v_l v_t \frac{(v_l^2 - v_t^2)^2}{(2v_l^3 + v_t^3)^2}\tau, \quad (7)$$

where

$$1/\tau = 1/\tau_R + 1/\tau_N. \quad (8)$$

τ_N is the normal-process relaxation time. The relaxation times τ^{11} and τ^1 correspond to processes affecting the local phonon streaming velocity, τ^{11} corresponding to phonon diffusion and τ^1 to coupling to the strain field. Niklasson shows that τ^{11} and τ^1 should be of the same order as τ. We set, therefore,

$$\tau^{11} = \tau \quad (9)$$

and treat τ^1/τ^{11} as an adjustable parameter. The relaxation times used are related to the phonon mean free paths by $\tau_U = \lambda_U/v_D$, λ_N/v_D, where v_D is the Debye velocity. Since τ_U is anisotropic, we have used an average value corresponding to a powder average. For v_l and v_t we have taken typical velocities at our densities obtained from

$$\rho v_l^2 = c_{11} \text{ and } \rho v_t^2 = \tfrac{1}{2}(c_{11} - c_{12}), \quad (10)$$

where c_{11} and c_{12} are the experimentally determined elastic constants.[2] These velocities correspond to the basal plane. The numerical values used for C_p/C_v, finally, were those given by Jarvis, Ramm, and Meyer.[8] In fitting our data by Eq. (3), we also have to make an assumption about the velocity v_{is} which is not accessible by experiment. v_{is} represents the velocity of first sound in the absence of coupling to the local temperature, and we assume for this the simple power law

$$v_{is} = v(0) - aT^n. \quad (11)$$

The exponent in here should be close to $n = 4$, but was actually chosen as the exponent of the temperature dependence of $C_p/C_v - 1$,

$$C_p/C_v - 1 = AT^n. \quad (12)$$

From the experimental data of Ref. 8, we obtain $n = 3.79$ at $p = 86$ bar and $n = 3.58$ at $p = 119$ bar. A least-squares fit of our data with Eq. (3) was then obtained, treating $v(0)$, a, and τ^1/τ^{11} as adjustable parameters. We allowed for small variation in τ_N and τ_U, but these changes were kept small and are probably not significant. We found, however, typically that τ^1 had to be increased above τ^{11} by amounts varying from $\tau^1/\tau^{11} = 1.9$ to $\tau^1/\tau^{11} = 45$ for various crystals and densities. Typical fits of this type are shown in Figs. 1 and 2 for the two densities investigated. As can be seen, the "knee" is farily well reproduced. The oscillations in the theoretical curve below the "knee" were never observed, but are almost

within the experimental uncertainty. The final rise in the curve represents the zero-sound regime; in this regine, however, Eq. (3) ceases to be valid. We also would like to comment on the excellent fit that was obtained above the knee, which justifies to some degree our choice of v_{is}. It should be realized that a more accurate theory would have to take into account the hexagonal symmetry of the crystal as well as the quantum solid character of solid helium. These calculations are at present not available.

The one outstanding difficulty in comparing Niklasson's theory with experiment is the large ratio τ^1/τ^{11} required. To some degree this may reflect the use of an isotropic version of the theory where proper hexagonal symmetry should have been used. We feel, in spite of this, that the effect is too large to be exclusively due to this source. We want to remark here that τ^1 corresponds to a length $l = v_2\tau^1$, where v_2 is the second-sound velocity, giving the range of the coupling between local phonon flow and lattice deformations. Our results seem to indicate that this range is much larger in solid helium than in a quasiharmonic solid with anharmonic corrections for which the theory of Niklasson was developed.

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[1]J. H. Vignos and H. A. Fairbank, Phys. Rev. 147, 185 (1966).
[2]R. H. Crepeau, O. Heybey, D. M. Lee, and S. A. Strauss, Phys. Rev. A 3, 1162 (1971).
[3]This effect was observed in quartz by J. S. Blinick and H. J. Maris, Phys. Rev. B 2, 2139 (1970).
[4]C. C. Ackerman and R. A. Guyer, Ann. Phys. (New York) 50, 128 (1968).
[5]E. M. Hogan, R. A. Guyer, and H. A. Fairbank, Phys. Rev. 185, 356 (1969).
[6]D. T. Lawson and H. A. Fairbank, J. Low Temp. Phys. 11, 363 (1973).
[7]G. Niklasson, Ann. Phys. (New York) 59, 263 (1970).
[8]J. F. Jarvis, D. Ramm, and H. Meyer, Phys. Rev. 170, 320 (1968).

Exact Solution of an Ising Model with Three-Spin Interactions on a Triangular Lattice

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The Ising model on a triangular lattice with three-spin interactions is solved exactly. The solution, which is obtained by solving an equivalent coloring problem using the Bethe *Ansatz* method, is given in terms of a simple algebraic relation. The specific heat is found to diverge with indices $\alpha = \alpha' = \frac{2}{3}$.

An outstanding open problem in lattice statistics has been the investigation of phase transitions in Ising systems which do not possess the up-down spin-reversal symmetry.[1,2] A well-known example which remains unsolved to this date is the Ising antiferromagnet in an external field. Another problem of similar nature that has been considered recently[3-5] is the Ising model on a triangular lattice with three-body interactions. This latter model is self-dual so that its transition temperature can be conjectured[3,6] using the Kramers-Wannier argument.[7] However, the nature of the phase transition has hitherto not been known.

We have succeeded in solving this model exactly. In this paper we report on our findings. It will be seen that the results are fundamentally different from those of the nearest-neighbor Ising models. While the final expression of our solution is quite simple, the analysis is rather lengthy and involved. For continuity in reading, therefore, we shall first state the result. An outline of the steps leading to the solution will also be given.

Consider a system of N spins $\sigma_i = \pm 1$ located at the vertices of a triangular lattice L. The three spins surrounding every face interact with a three-body interaction of strength $-J$, so that the Hamiltonian reads

$$\mathcal{H} = -J\sum \sigma_i\sigma_j\sigma_k, \tag{1}$$

with the summation extending over all faces of L. Let Z be the partition function defined by (1). We find the following expression for $Z^{1/N}$ in the