Nonlinear-Evolution Equations of Physical Significance*

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We present the inverse scattering method which provides a means of solution of the initial-value problem for a broad class of nonlinear evolution equations. Special cases include the sine-Gordon equation, the sinh-Gordon equation, the Benney-Newell equation, the Korteweg–de Vries equation, the modified Korteweg–de Vries equation, and generalizations.

One of the most exciting recent advances in applied mathematics and theoretical physics has been a method of solution for the initial value problem for certain nonlinear partial differential equations which arise naturally in many scientific areas.[1-4] There are four key steps in the method of solution. (1) First, set up an appropriate, linear scattering (eigenvalue) problem in the "space" variable where the solution of the nonlinear evolution equation plays the role of the potential. (2) Choose the "time" dependence of the eigenfunctions in such a way that the eigenvalues remain time invariant as the potential evolves according to the evolution equation. Although it is yet to be proved, the ability to achieve this step appears to depend on the existence of an infinite number of independent conservation laws for the evolution equation. (3) Solve the direct scattering problem at the initial "time" and determine the "time" dependence of the scattering data. (4) Do the inverse scattering problem at later "times"; namely, knowing the (discrete) eigenvalues corresponding to the bound states and knowing the time dependence of the other scattering data, reconstruct the potential. The final step can be written in terms of a linear integral equation (or a coupled set of linear integral equations) from which one can compute the solution to the evolution equation for all time.

We have found that many nonlinear evolution equations can be solved by the following scattering problem:

$$\partial v_1/\partial x + i\zeta v_1 = q(x,t)v_2,$$
$$\partial v_2/\partial x - i\zeta v_2 = r(x,t)v_1. \tag{1}$$

Choose the time dependence of the eigenfunctions v_1 and v_2 to be

$$\partial v_1/\partial t = A(x,t,\zeta)v_1 + B(x,t,\zeta)v_2,$$
$$\partial v_2/\partial t = C(x,t,\zeta)v_1 - A(x,t,\zeta)v_2. \tag{2}$$

The eigenvalues ζ are time invariant when

$$\partial A/\partial x = qC - rB,$$
$$\partial B/\partial x + 2i\zeta B = 2q/\partial t - 2Aq, \tag{3}$$
$$\partial C/\partial x - 2i\zeta C = \partial r/\partial t + 2Ar.$$

Equations (3) are obtained by cross differentiation of the systems (1) and (2). Finite expansions of A, B, and C in terms of the parameter $2i\zeta$ allow us to determine the class of evolution equations which can be solved by the inverse scattering method. It can be verified that the following evolution equations belong to this class.

Class I.—Take $A = -4i\zeta^3 - 2iqr\zeta + r\partial q/\partial x - q\partial r/\partial x$ and find

$$\partial q/\partial t - 6rq\partial q/\partial x + \partial^3 q/\partial x^3 = 0,$$
$$\partial r/\partial t - 6rq\partial r/\partial x + \partial^3 r/\partial x^3 = 0. \tag{4}$$

When $r=-1$, (4) reduces to the Korteweg–de Vries (KdV) equation, and the system of equations (1) reduces to the Schrödinger equation[1]

$$\partial^2 v_2/\partial x^2 + [\zeta^2 + q(x,t)]v_2 = 0. \tag{5}$$

When $r = \pm q$, (4) reduces to the modified KdV equation[5]

$$\partial q/\partial t \mp 6q^2\partial q/\partial x + \partial^3 q/\partial x^3 = 0. \tag{6}$$

When $r = +q$, q real, the eigenvalue problem posed by (1) is self-adjoint, and hence all eigenvalues are real. In this case no solitons arise, and the final state can be shown to decay algebraically in time.[6] When $r=-q$ a class of paired permanent waves (with complex eigenvalues) arise in addition to the individual solitons.

Class II.—Take $A = a/\zeta$, and find

$$\partial a/\partial x = \tfrac{1}{2}i\partial(qr)/\partial t, \quad \partial^2 q/\partial x\partial t = -4iaq,$$
$$\partial^2 r/\partial x\partial t = -4iar. \tag{7}$$

When $a = \frac{1}{4}i\cos u$, $r = -q = \frac{1}{2}\partial u/\partial x$, the sine-Gordon equation

$$\partial^2 u/\partial x\partial t = \sin u \tag{8}$$

is obtained. Equations (6) and (8) are solved by the same scattering problem.[3] In addition to the traveling kink and antikink solutions, the only other localized stable solutions are soliton states (breathers) which oscillate in time and which correspond to paired complex eigenvalues. The eigenvalues corresponding to the modes of a given breather in its own rest frame all lie on the circle $\zeta\zeta^*=\frac{1}{4}$. In its rest frame, a particular breather solution is

$$u(x,t)=4\tan^{-1}\{[(1-\omega^2)/\omega^2]^{1/2}\cos\omega(T-T_0)\times\operatorname{sech}[1-\omega^2)^{1/2}(X-X_0)]\},\qquad(6)$$

where $\omega=-2\operatorname{Re}\zeta$ and $x=\frac{1}{2}(X+T)$, $t=\frac{1}{2}(X-T)$. These solutions have been obtained by Lamb[7] and Seeger, Donth, and Kochendorfer[8] from the Bäcklund transformation and have been observed numerically.[9,10]

If $a=\frac{1}{4}i\cosh u$, $r=q=\frac{1}{2}\partial u/\partial x$, the sinh-Gordon equation

$$\partial^2u/\partial x\partial t=\sinh u\qquad(10)$$

is obtained. Since the eigenvalue equation is self-adjoint, no solitons arise and the final state will decay in time.

Class III.—Take $A=-2i\zeta^2-irq$ and find

$$i\partial q/\partial t+\partial^2q/\partial x^2-2q^2r=0,$$
$$i\partial r/\partial t-\partial^2r/\partial x^2+2qr^2=0.\qquad(11)$$

In the special case $r=-q^*$ $(+q^*)$, this corresponds to the equation describing the evolution of the envelope of an almost monochromatic wave[11,12] when the Benjamin-Feir instability[13,14] is operative (inoperative). The case $r=-q^*$ has been solved by Zakharov and Shabat.[2] When $r=q^*$, the eigenvalue problem (1) is again self-adjoint and thus no solitons arise. The solution decays algebraically in time and has a self-similar structure.[6]

These equations (4), (7), and (11) are special cases of the general class of evolution equations which can be derived and solved by our procedure. The generalized Korteweg–de Vries equation[15] can also be obtained by this procedure when $r=1$.

The direct and inverse scattering analysis will be published elsewhere. Here we merely quote the result.

Given $q(x,0)$, $r(x,0)$ sufficiently smooth and decaying sufficiently fast as $|x|\to\infty$, the solutions $q(x,t)$, $r(x,t)$ for all time are given by

$$q(x,t)=-2K_1(x,x),$$
$$r(x,t)=-2\bar{K}_2(x,x),\qquad(12)$$

where

$$K(x,y)-\begin{bmatrix}1\\0\end{bmatrix}\bar{F}(x+y)-\int_x^\infty\bar{K}(x,s)\bar{F}(s+y)\,ds=0,$$
$$\bar{K}(x,y)+\begin{bmatrix}0\\1\end{bmatrix}F(x+y)+\int_x^\infty K(x,s)F(s+y)\,ds=0,\qquad(13)$$

and

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\xi,t)}{a(\xi)}e^{i\xi x}d\xi-i\sum_k C_k\exp(i\zeta_kx),$$
$$\bar{F}(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{b}(\xi,t)}{\bar{a}(\xi)}e^{-i\xi x}d\xi+i\sum_k\bar{C}_k\exp(-i\bar{\zeta}_kx),\qquad(14)$$

and

$$K(x,y)=\begin{bmatrix}K_1(x,y)\\K_2(x,y)\end{bmatrix},\quad\bar{K}=\begin{bmatrix}\bar{K}_1(x,y)\\\bar{K}_2(x,y)\end{bmatrix}.$$

The ζ_k $(\bar{\zeta}_k)$ are the eigenvalues of (1) which lie in the upper (lower) half plane; the a, b, C_k, $\bar{a}$, $\bar{b}$, $\bar{C}_k$ are the scattering data and have the time dependences

$$a(\xi)=a_0(\xi),\quad\bar{a}(\xi)=\bar{a}_0(\xi),$$
$$b(\xi,t)=b_0(\xi)\exp[-2A_0(\xi)t],$$
$$\bar{b}(\xi,t)=\bar{b}_0(\xi)\exp[2A_0(\xi)t],\qquad(15)$$
$$C_k=C_{k0}\exp[-2A_0(\zeta_k)t],$$
$$\bar{C}_k=\bar{C}_{k0}\exp[2A_0(\bar{\zeta}_k)t],\text{ and }A_0(\zeta)=\lim_{|x|\to\infty}A(x,\zeta;\zeta).$$

The eigenvalues and the various constants are determined by solving the eigenvalue problem (1) at the initial time. Following Zakharov and Shabat,[2] an infinite set of conservation laws can be found for the above systems and correspond in the case of Eq. (6) (also for KdV) to the polynomial conserved densities of integer rank.[6]

We note also that a large class of linear problems can be solved by this method. For such cases take $r\equiv0$, and the procedure reduces to the Fourier-transform approach. The conservation laws for these cases are trivial.

The solution of the system of linear integral equations can readily be found in closed form when the reflection coefficients $b_0(\xi)$ and $\bar{b}_0(\xi)$ are identically zero. This requires a very special class of initial conditions. For arbitrary initial conditions, (13) can be solved asymptotically (large t) following Ref. 6. The general asymptotic solution is essentially a sequence of kinks (solitons) and breathers (paired solitons with periodic behavior) superposed on a decaying background of oscillatory structure. The latter corresponds to the continuous spectrum.

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Energy Spectrum of Nuclei with $Z \geqslant 60$ as Evidence for a New Source of Cosmic Rays*

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We have found the energy spectrum of cosmic rays with $Z \geqslant 60$ to be much steeper in the neighborhood of 1 GeV/amu than measured spectra of less massive cosmic rays. The data analysis includes effects due to solar modulation. This low-energy enhancement of high-Z primaries implies sources which either are strongly enriched in very heavy elements or have a Z-dependent acceleration mechanism.

In September of 1970, we carried out a balloon flight of a 22-m^2 passive detector array in order to study trans-iron cosmic rays. Included in the array were one layer of 200-μm G-5 nuclear emulsion, one layer of fast-film Cherenkov detectors, and forty sheets of Lexan plastic track detectors. The effective exposure for extremely heavy primaries was ~60 h at a mean atmospheric depth of ~3.7 g/cm^2. A detailed description of the flight and data analysis has been presented elsewhere.[1]

One of the principal goals of the experiment was to determine for the first time an energy spectrum for cosmic rays with $Z \geqslant 60$. The realization of this goal revealed a spectrum much steeper than those of lighter nucleonic cosmic rays. We consider this feature to be very significant in terms of implications for cosmic-ray sources. All of our 35 events with $Z \geqslant 60$ were found by scanning 17.8 m^2 of the emulsion with stereomicroscopes. The nature of the spectrum dictated that we eliminate insofar as possible any uncertainties associated with scanning efficiency. To this end, the entire area was completely rescanned by different observers with the result that no new events were found. In addition, an