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Scaling Function for Critical Scattering

Michael E. Fisher and Amnon Aharony
Baker Laboratory, Cornell University, Ithaca, New York 14850
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The zero-field, two-point correlation function of an n-vector system in $d=4-\epsilon$ dimensions is calculated to order ϵ^2 for $T \geqslant T_c$. The scaling function is obtained as a closed, cutoff-independent integral. As $t=(T-T_c)/T_c \to 0$ at fixed wave vector q, the leading variation is $\hat{E}_1^{n,d}(q)t^{1-\alpha}+\hat{E}_2^{n,d}(q)t$, where α is the specific-heat exponent; thence the maximum in the scattering above T_c is located, in good agreement with high-T series-expansion estimates.

As a critical point is approached in zero field (i.e., along a line of symmetry) the correlation length diverges[1,2] as $\xi(T)\approx f_1 a t^{-\nu}$ where $t=(T-T_c)/T_c$, and a is the lattice spacing. The exponent ν depends on d, the dimensionality of the lattice, and on n, the number of "spin" components. Similarly, the scattering intensity, which is proportional to the Fourier transform, $\hat{G}(\vec{q}, T)$, of the two-point correlation function becomes large at low wave numbers $\vec{q}$.

According to the scaling hypothesis[1-4] for the critical correlations, one can write

$$\hat{G}(\vec{q}, T)\approx Ct^{-\gamma}\hat{D}(q^2\xi^2), \tag{1}$$

for $t \ll 1$ and $qa \ll 1$. On adopting the normalizations $\hat{D}=-d\hat{D}/dy=1$ at $y=0$, the constant C becomes the amplitude of the (reduced) initial susceptibility or zero-angle scattering function $\hat{\chi}_0(T)=\hat{G}(0,T)$, which diverges with exponent $\gamma=\gamma(n,d)$. Similarly, $\xi\equiv\xi_1$ must then be identified as the second-moment correlation length.[1,4] The amplitudes C and f_1 must depend on the details of the interactions, the lattice structure, etc., but the form of the scaling function $\hat{D}(x^2)$ is expected to be "universal," depending only on n and d.[5,6]

In this note we present, for the first time, an analytic calculation of the scaling function $\hat{D}(x^2)$ exact to order ϵ^2, where $d=4-\epsilon$,[7,8] for a system of general n with isotropic ($\vec{s}\cdot\vec{s}'$) interactions of finite range. Our result may be written

$$1/\hat{D}(x^2)=1+x^2-4p_n x^4 Q(x^2)\epsilon^2+O(\epsilon^3), \tag{2}$$

where $p_n=\frac{1}{2}(n+2)/(n+8)^2$ and $Q(x^2)$ is defined by the fully convergent integral

$$Q(y)=y^{-2}\int_0^\infty dz\,[z(1+\tfrac{1}{4}z)]^{1/2}\ln[(1+\tfrac{1}{4}z)^{1/2}+\tfrac{1}{2}z^{1/2}] \times\{(1+z)^{-1}-y(1+z)^{-3}-\tfrac{1}{2}y^{-1}z^{-1}[1+y+z-(1+2y+2z+y^2-2yz+z^2)^{1/2}]\}. \tag{3}$$

The cutoff independence of this integral (see below) confirms the expected universality of $\hat{D}(x^2)$. In the limits $\epsilon\to 0$ and $n\to\infty$ the result reduces to the Ornstein-Zernike (OZ) form, $\hat{D}=(1+x^2)^{-1}$, as may be expected. (A separate exact calculation to order[9] $1/n$ has been undertaken.[10]) To leading order in ϵ the deviation from the OZ form is proportional to the exponent[7,8] $\eta=\eta(n,d)$, which determines the critical-point decay of correlation.[1,2]

In the low-momentum limit ($q\to 0$ at fixed $T>T_c$), expansion of (3) in powers of y leads to the small-x expansion $1/\hat{D}(x^2)=1+x^2-\Sigma_4x^4+\Sigma_6x^6-\dots$, the form of which has been anticipated on general grounds.[1,2,4] Numerical evaluation of the definite integrals derived from (3) yields $\Sigma_{2k}=2b_{2k}p_n\epsilon^2+O(\epsilon^3)$, with b_4

$\simeq 0.007\,520$ and $b_6 \simeq 0.000\,191\,9$. The surprisingly small values of the coefficients b_{2k} explain the experimentally observed fact[11-13] that the OZ form for $\hat{D}$ is a rather good approximation for small x (i.e., low q at fixed t) when $d=3$. As evident from Table I, these results also confirm closely the comparatively imprecise estimates of Σ_4 which can be derived from high-temperature series.[4,14-16]

The finite-momentum behavior, i.e., $T \to T_c$ at fixed $q>0$ (but $qa \ll 1$), following from (5) is of considerable interest. In contrast to the scaling functions previously calculated for the equation of state,[17,18] the correlation scaling function *itself* is expected to display intrinsic, critical, (n, d)-dependent singularities for large x. This follows from the conclusion[19] that the scattering in the finite momentum limit should vary as

$$\hat{G}(q, T) \approx \hat{D}_0/(aq)^{2-\eta} + \hat{E}_1(q) t^{1-\alpha} + \hat{E}_2(q) t + \dots, \tag{4}$$

where $\alpha = \alpha(n, d)$ is the specific-heat exponent. This form has played a valuable role in the interpretation of critical-resistivity experiments.[20] It is consistent with the series-expansion data[4,14-16] although the coefficients $\hat{E}_1$ and $\hat{E}_2$ cannot be determined with any precision. By comparison with (1), one conjectures[19] that

$$\hat{D}(x^2) \approx D_0^{\infty}/x^{2-\eta} + D_1^{\infty}/x^{2-\eta+(1-\alpha)/\nu} + D_2^{\infty}/x^{2-\eta+1/\nu} + \dots \tag{5}$$

as $x \to \infty$, where $D_0^{\infty}(n, d) = (f_1^{2-\eta}/C)\hat{D}_0 = \varphi_c^{\eta}/(1+\frac{1}{2}\eta\varphi_c^2)$, in which φ_c is the universal parameter entering the Fisher-Burford approximant for $\hat{D}(x^2)$.[4] From (5) one obtains the scaling prediction $\hat{E}_1^{n,d}(q) \sim q^{-2+\eta-(1-\alpha)/\nu}$ and an analogous prediction for $\hat{E}_2^{n,d}(q)$.[19,15] The conjecture (5) may at last be checked analytically by studying the behavior of $Q(y)$ for large y. A somewhat tricky calculation[21] leads to

$$Q(y) \approx Q_0' \ln y/y - Q_0/y + Q_1''(\ln y)^2/y^2 + Q_1' \ln y/y^2 - Q_1/y^2 + \dots, \tag{6}$$

with $Q_0' = \frac{1}{8}$, $Q_1'' = \frac{3}{8}$, $Q_1' = 0$, $Q_0 \simeq 0.517\,76$, and $0 < Q_1 < 0.013$. A careful comparison with the ϵ expansion of (5) to second order, using the known exponent expansions,[7,8] then reveals that the values of Q_0' and Q_0'' are precisely consistent with the anticipated singularity structure. (Some remarkable algebraic cancellations occur![21]) Furthermore we obtain

$$\begin{aligned} &D_0^{\infty} = 1 - 4Q_0 p_n \epsilon^2 + O(\epsilon^3), \quad D_1^{\infty} = \frac{n+2}{4-n} + \frac{(n+2)(7n+20)}{(n+8)(4-n)^2}\epsilon + O(\epsilon^2), \\ &D_2^{\infty} = -\frac{6}{4-n} - \frac{(n+2)(7n+20)}{(n+8)(4-n)^2}\epsilon + O(\epsilon^2). \end{aligned} \tag{7}$$

Table I compares the truncated expansion for D_0^{∞} with series expansion estimates for $d=3$. The apparent precision is about 5%, but it must be borne in mind that the uncertainties attached to the series estimates may not fully reflect the exponent uncertainties. It is remarkable that although the deviation

TABLE I. Comparison of numerical estimates for three dimensions.

Parameter	n	Order ϵ^2	High-T series
η	1	0.019 [a]	0.056 ± 0.008 [b]
	3	0.021 [a]	0.043 ± 0.014 [c]
Σ_4	1	2.8×10^{-4}	$\sim 6.3 \times 10^{-4}$ [d]
	3	3.1×10^{-4}	$\sim 2.6 \times 10^{-4}$ [d]
$D_0^{\infty} \simeq \varphi_c^{\eta}$	1	0.962	0.90 ± 0.01 [c]
	3	0.957	0.91 ± 0.02 [c]
$\tau_{\max} = \tau_{\max}/(f_1 aq)^{1/\nu}$	1	1.3×10^{-2}	$(1.0-2.3) \times 10^{-2}$ [c]
	3	1.8×10^{-2}	$(1-4) \times 10^{-2}$ [c]

[a]Ref. 8.
[b]Ref. 4.
[c]Ref. 16.
[d]Using the Fisher-Burford approximant (Refs. 4 and 16), one has $\Sigma_4 \simeq \frac{1}{2}\eta\varphi_c^2$; direct series estimates for Σ_4 are less precise (Ref. 21).

of $\hat{D}(x^2)$ from the OZ form is only of order ϵ^2, the coefficients D_1^∞ and D_2^∞ have contributions of both first order and zeroth order. [As $\epsilon \to 0$ or $n \to \infty$, one has $\eta \to 0$ and $1-\alpha \to 2\nu$ so that the two correction terms in (5) combine to give the OZ result $-1/x^4$.] The divergence of the coefficients D_1^∞ and D_2^∞ at $n=4$ is associated with the vanishing of α near $n=4$ as $\epsilon \to 0$, which, in turn, corresponds to a logarithmic singularity in the specific heat. A closer investigation confirms this interpretation and yields the logarithmic amplitude.

By putting $t/(f_1 aq)^{1/\nu} = \tau$, the finite-momentum results can be cast, to order ϵ^2, in the transparent form

$$\hat{G}(q, T)/\hat{G}_c(q) \approx 1 + (\gamma - 1)\tau(\tau^{-\alpha} - 1)/\alpha - \tau + \dots, \tag{8}$$

where the limit $\alpha \to 0$ correctly yields a logarithmic factor. From this expression, by neglecting the higher-order powers of τ, one may locate a maximum in the scattering above T_c at $\tau_{\max} \simeq [(1-\alpha)(\gamma-1)/(\gamma - 1 + \alpha)]^{1/\alpha}$. This maximum was first predicted numerically by series expansions,[4,16] but its precise location was not at all accurately revealed. The good agreement between this estimate of $\tau_{\max}$ and those derived from series (see Table I) is very gratifying and considerably strengthens the interpretation of the maxima seen experimentally in the same vicinity.[12,13] For $\alpha > 0$, the existence of a maximum can already be predicted from (4), but its continued presence for $\alpha < 0$ (for all finite n) could not previously be understood theoretically.

The derivation[21] of the result (2) uses the graphical ϵ-expansion technique[8,17] for the standard, continuous spin, US,[4] Hamiltonian[8] with a sharp cutoff at $|\vec{q}| = \pi\Lambda$ (a being set equal to unity). The correlation function is expanded in powers of u with propagator $G_0(\vec{q}, r) = (r + q^2)^{-1}$, where r is chosen to be the inverse susceptibility $1/\hat{\chi}_0 \approx t^\gamma/C$. Using Wilson's fixed-point value[8] for $u_{0c}(\epsilon)$, the result is

$$\hat{G} = G_0[1 + 4p_n\epsilon^2 q^2 G_0 K + O(\epsilon^3)], \tag{9}$$

where

$$K(\vec{q}, r) = \tfrac{1}{4}\pi^{-4}q^{-2}\int d^4q' \int d^4q''\, G_0(\vec{q}'', r) G_0(\vec{q}' + \vec{q}'', r)[G_0(\vec{q}' - \vec{q}, r) - G_0(\vec{q}', r)]. \tag{10}$$

For small r one finds $K(0, r) \approx \frac{1}{8}\ln(r/\Lambda^2) - k + O(r \ln r)$, where the constant $k \simeq 0.090\,857$ depends on the form of the cutoff. It may be used in deriving the ϵ expansion of the nonuniversal amplitude C and f_1. Note that this is a first step towards the ultimate calculation of the magnitude of the corrections to scaling for specific model systems. (A sharp cutoff is appropriate in this context.) Next one sets[22] $\vec{q} = \vec{x}/\xi_1 = c r^{1/(2-\eta)}\vec{x}$, where $c = C^{1/(2-\eta)}/f_1$, and, by comparison with (1), calculates $\hat{D}(x^2)$ from the $r \to 0$ limit of $r\hat{G}(\vec{x}cr^{1/(2-\eta)}, r)$. Making ϵ expansions of $\hat{D}$ and c, comparing coefficients with those arising from (9), and using the adopted normalization then leads to

$$Q(x^2) = x^{-2}\lim_{r \to 0}[K(\vec{x}\sqrt{r}, r) - K(0, r)]. \tag{11}$$

This may finally be transformed into the form (3). In the process the result is shown to be cutoff independent.

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Experimental Observation of the Nonoscillatory Parametric Instability at the Lower-Hybrid Frequency

R. P. H. Chang
Bell Laboratories, Murray Hill, New Jersey 07974

and

M. Porkolab
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540
(Received 7 August 1973)

The nonoscillatory parametric instability is observed when the "pump" frequency is near the lower-hybrid frequency. Strong heating of both the ions and electrons has also been measured.

Although predicted by theory,[1] to our knowledge the nonoscillatory parametric instability (the oscillating two-stream instability or the purely growing mode) has not yet been verified experimentally. Some of the reasons for the difficulty of observing such modes are the following: (a) The oscillatory decay instability has a lower threshold in most cases of interest than the nonoscillatory instability; thus it tends to "cover up" the possible presence of the latter.[1] (b) Spectral analysis and wavelength measurements are very difficult since the real part of the frequency for the purely growing mode is zero. Recently it has been shown, however, that for pump frequencies very near the lower-hybrid frequency the threshold for the oscillatory parametric decay instability increases rapidly.[2] The reason for such increase in the threshold is that the wave-vector matching condition produces considerable electron Landau damping of the ion