spectrum just discussed, and He^+ ions of known kinetic energy, unambiguous answers to these questions can be given.

The He^+ kinetic-energy spectrum ($\Theta = 8°$), or rather the energy-loss spectrum, which directly shows the He^+ intensity as a function of the inelastic energy loss, is given in Fig. 4. We have now searched by delayed coincidence for those energy losses which belong to certain features of the electron spectrum. For three examples the result is shown in Fig. 4. In each case the number of true coincidences, normalized to the He^+ intensity in the loss spectrum, is plotted as a function of energy loss. It is found that (i) the energy loss Q which belongs to the strongest electron peak at 23.4 eV is $Q = 45.0$ eV; (ii) the energy loss belonging to the peak at 28.5 eV is $Q = 50$ eV; and (iii) the energy loss belonging to the low-energy continuous part of the electron spectrum around 1.5 eV is $Q = 23$ eV.

The results (i) and (ii) prove that the electron peaks are, indeed, due to process (1), with Ne** states as indicated in Fig. 3, and He^+ remaining in the ground state. Result (iii) proves that the continuous background is "real," and due to molecular autoionization into a state dissociating into He^+ and Ne^+ in their ground states, i.e., to the process

$$He^+ + Ne \rightarrow (He\text{-}Ne)^{++}_{\text{small } R} + e^-(1.5 \text{ eV}) \quad \rightarrow He^+(1s) + Ne^+((2p^5)\,^2P). \tag{3}$$

Results of more detailed studies will be published later.

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Autoresonant Accelerator Concept

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We present a new ion accelerator idea which combines the basic concepts of traveling wave and collective acceleration. The lower cyclotron mode of a relativistic electron beam along a spatially decreasing magnetic field B constitutes the traveling wave, the wave phase velocity varying inversely with B. The negative energy character of the wave allows automatic extraction of energy from the electron beam for the acceleration process. Typical ion energy versus current outputs are estimated.

Conventional traveling-wave accelerators use the electric field of a traveling electromagnetic eigenmode of a (generally) vacuum wave guide to effect the acceleration of charged particles which are in resonance with the wave. Spatially structured wave guides are used to increase the phase velocity of the wave as it moves along the accelerator, and energy is supplied to the wave from an external rf power source. Because of technological limitations on rf power handling and the maximum electric field that can be sustained within the wave-guide structure, the accelerating electric fields of such accelerators are limited to well under 1 MV/cm.

By contrast, in the collective-accelerator concept,[1-7] the collective electric field of a relatively dense bunch of electrons provides the accelerating electric field for a relatively less dense bunch of ions. In this concept accelerating fields larger than 1 MV/cm appear achievable, allowing the possibility of much more compact ion accelerators.

The use of traveling waves in a collective fashion has been suggested by many authors.[5-7] The autoresonant accelerator (ARA) discussed below is a particular embodiment of this combination of traveling-wave and collective accelerator concepts. In the ARA the medium in which the wave travels is a relativistic electron beam immersed in a dc magnetic field. The traveling wave is the Doppler-shifted cyclotron mode of the relativistic electron beam and has the property that its phase velocity increases with decreasing magnetic field. As in the usual traveling-wave accelerator, in order to accelerate ions which are in resonance with the wave, the phase velocity of the wave must be adiabatically increased from a low phase velocity at the input end of the accelerator to a higher phase velocity at the output end of the accelerator. For this cyclotron wave, this increasing phase velocity can be achieved by having the dc magnetic field decrease spatially in a prescribed manner from the input end of the accelerator to the output end.

A novel feature of this accelerator concept lies in the fact that the relativistic electron beam is more than simply a medium for the propagation of the wave: It is an active medium which, as shown below, automatically serves as the power source both for reinforcing the electric field of the traveling wave and for accelerating the ions. Thus, in principle, the ARA has the capability of large (greater than 1 MV/cm) accelerating electric fields while at the same time eliminating the power handling problems associated with high-power external rf sources. Since moderate-size relativistic-e-beam diodes typically produce hundreds of kiloamperes in the several-MeV energy range with a pulse time of roughly 50–100 nsec, ample power flow (10^{11}–10^{12} W) is available for the acceleration of hundreds of amperes of multi-GeV ions during the e-beam pulse.

In the remainder of this Letter we describe the general features of the ARA concept: the electron beam and magnetic field configuration with its attendant equilibrium and stability requirements, the characteristics of the Doppler-shifted cyclotron mode, the equations governing the energetics of the e beam, traveling wave, and accelerated ions, along with estimates of typical output parameters.

It is well known that an electron beam may be stably propagated in a vacuum interior to a conducting, cylindrical guide along a constant magnetic guide field, as shown in Fig. 1, if the following sufficient conditions are met for the existence of such an equilibrium[8,9]:

$$\omega_p^2 < \gamma\Omega c/a, \tag{1}$$

$$2\omega_p^2 < \gamma^2\Omega^2, \tag{2}$$

$$\omega_p^2 a^2 < 4c^2; \tag{3}$$

and the stability thereof[10]:

$$\omega_p^2 < \gamma\Omega c/b, \tag{4}$$

where $\omega_p^2 \equiv 4\pi ne^2/\gamma m$ and $\Omega \equiv eB/\gamma mc$. Here n is the electron beam density as seen in the lab frame; e, the absolute value of the electron charge; m, the electron rest mass; B, the magnetic guide field; a, the beam radius; b, the radius of the conducting guide; and c, the speed of light. The electron relativistic factor γ is considered large compared to unity ($\gamma \gg 1$), so that the electron flow velocity $v_e \approx c$. In the following discussion, we shall also restrict our attention to systems where $a \simeq b$.

The equations of equilibrium, which are rigorously sufficient when the inequality is strongly satisfied, have a simple physical interpretation. Equation (1) requires that the radial self-electric field of the electron beam be smaller than the confining longitudinal magnetic field, when viewed from the rest frame of the electron beam, while Eq. (2) requires simply that the centrifugal force associated with the $\vec{E}\times\vec{B}$ rotation of the beam be smaller than the $\vec{v}\times\vec{B}$ restoring force. Equation (3) is equivalent to the usual Lawson criteria. The sufficient condition for stability is an extension of work by Davidson and Krall[11]

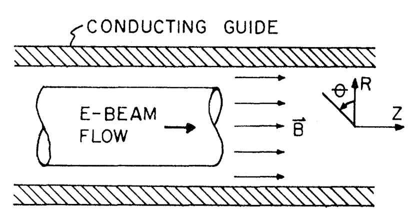

FIG. 1. Equilibrium electron-beam and magnetic field configurations.

on rigid-rotor equilibria.

The allowed axisymmetric eigenmodes for this cylindrically symmetric, longitudinally homogeneous system exhibit electric field components of the form

$$E_z \propto J_0(\kappa_\perp r)\exp[i(\kappa_z z-\omega t)], \quad E_r, E_\theta \propto J_1(\kappa_\perp r)\exp[i(\kappa_z z-\omega t)],$$

with the resultant dispersion relation

$$\epsilon(\omega,\vec{\kappa}) = (\omega-\kappa_z v_e)^2(\omega^2-\omega_p^2-\kappa^2c^2)^2[(\omega-\kappa_z v_e)^2-\omega_p^2/\gamma^2] - \Omega^2(\omega^2-\kappa^2c^2)[(\omega-\kappa_z v_e)^2(\omega^2-\kappa^2c^2)-(\omega_p^2/\gamma^2)(\omega^2-\kappa_z^2 v_e^2)]=0, \quad (5)$$

where $\vec{\kappa}=\kappa_\perp\hat{r}+\kappa_z\hat{z}$. The functions J_0 and J_1 are the usual Bessel functions of the first kind of zero and first order, respectively. Here r,θ,z are the usual cylindrical coordinates as depicted in Fig. 1.

An examination of the eight eigenmodes available, as exhibited in Fig. 2, under the restrictions imposed by Eqs. (1)–(4), shows that only one eigenmode, the Doppler-shifted lower branch of the upper-hybrid mode, has a phase velocity $v_{\rm ph}$ variable from a very low value, $v_{\rm ph}\ll c$, to a value approaching $v_e\approx c$. Using Eq. (5), the dispersion relation of this mode is found to be roughly that of the (Doppler-shifted) lower cyclotron mode and is given by

$$\omega=\kappa_z v_e-\Omega. \quad (6)$$

If, instead of a longitudinally homogeneous system (homogeneous in z), we allow an adiabatic spatial decrease in B in the z direction, the wave vectors of the eigenmodes will adiabatically change in order to continue to satisfy the dispersion relation, the frequency of each mode remaining constant. In particular, the phase velocity of the lower cyclotron mode is found to be

$$v_{\rm ph}(z)=v_e\omega_0/[\omega_0+\Omega(z)], \quad (7)$$

where ω_0 is the constant frequency of the mode. Thus, if ions are placed in the potential trough of this wave at a point where the magnetic field is high and the phase velocity low and if the magnetic field decreases in an appropriate manner in the direction of propagation of the wave (and of the electron beam), then the ions can be accelerated to a velocity approaching $v_e\approx c$, and hence to energies in the multi-GeV range.

In such a system, the electrons will tend to stay on magnetic flux surfaces and thus expand, as B decreases, in a flux-preserving manner. Hence, the walls of the guide tube are also expanded in a similar manner, i.e., $b\approx a\propto B^{-1/2}$. The spatial configurations of the magnetic field, electron beam, and wave guide are indicated in Fig. 3. Equation (6) governs the adiabatic change in $\kappa_z(z)$ as a function of $\Omega(z)$. The equation governing the adiabatic spatial change in electric field strength in such a system is the usual equation for convection of wave energy:

$$\nabla\cdot[\vec{v}_g\,\omega(\partial\epsilon/\partial\omega)E^2/8\pi+\tfrac{1}{2}n_i M v^2\vec{v}]=0. \quad (8)$$

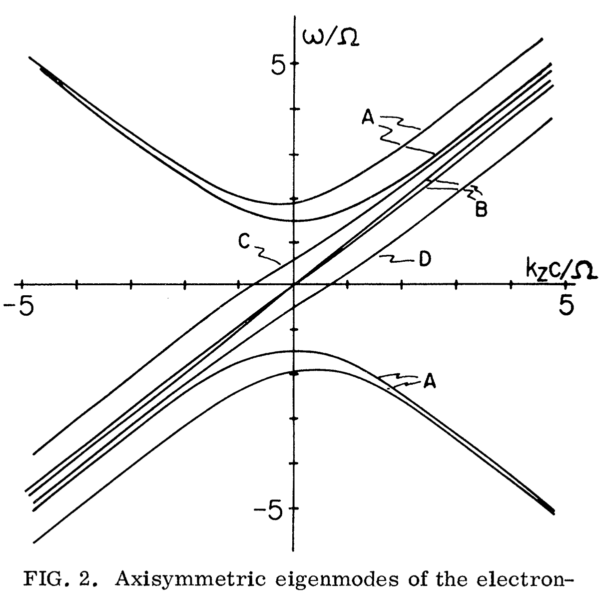

FIG. 2. Axisymmetric eigenmodes of the electron-beam–magnetic-field configuration. A identifies the four modified electromagnetic modes; B, the two Doppler-shifted plasma (ω_{pe}) modes; C, the Doppler-shifted upper branch of the upper hybrid mode; and D, the Doppler-shifted lower branch of the upper-hybrid mode.

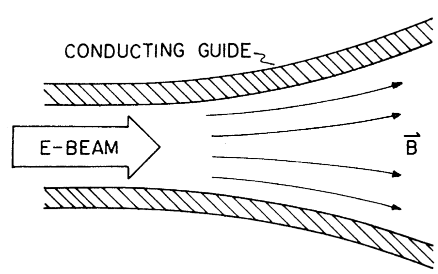

FIG. 3. System configuration of the ARA.

Here v_g, the group velocity of the lower cyclotron mode, is given by $v_g = v_e \hat{z}$, while $\dot{z} = \dot{z}_{\mathrm{ph}} \hat{z}$ is the instantaneous velocity of the accelerating ions; n_i is the ion density, and M is the ion mass. (Thermal corrections to the ion-energy term are negligible, as one can easily show that the ions are adiabatically cooled in the potential well during the acceleration process.) Thus, the sum of the fluxes of wave energy, $\omega(\partial\epsilon/\partial\omega)E^2/8\pi$, and energy in the resonant ions is conserved. In conventional traveling-wave accelerators using electromagnetic modes, the wave energy is positive ($\omega\, \partial\epsilon/\partial\omega > 0$), so that any net increase in energy of the ions results in a net loss of energy in the electric field, necessitating the continual replenishing of energy from external sources to keep the traveling wave and acceleration process intact. This is not the case for the lower cyclotron mode: It is a negative-energy wave, as can be determined either by a simple calculation or by noting that by use of a Lorentz transformation from the electron beam frame (where all waves are positive-energy waves) to the lab frame, ω changes sign as a result of the Doppler effect, thus changing the sign of the wave energy, the sign of $\partial\epsilon/\partial\omega$ being frame invariant. This means that a net increase in energy of the ions results in a net *increase* in electric field energy flux. The fundamental reason for this difference lies in the fact that the ARA concept utilizes an active propagation medium, the relativistic electron beam, with available free energy. During acceleration, energy is drained out of the longitudinal motion of the electron beam both to accelerate the ions and to reinforce the field of the traveling lower cyclotron wave. No other energy source for the acceleration process is required.

The applicability of the foregoing analysis rests on the requirement that the momentum and spatial perturbations of the electron beam be small. Thus, the potential of the wave must never be so large that trapping of the electrons will occur, and the radial excursion of the electron beam must be small compared to a. Coupled with the requirement of total energy conservation, Eqs. (1)–(8), and conservation of electron and ion current, these restrictions form a sufficient set to estimate total output parameters that would be achievable in such an accelerator. For purposes of illustration, we consider a large electron-beam diode device with output parameters of 12.5 MV at 100 kA. (As a practical comparison, each of the four units comprising the Aurora facility[12] at Harry Diamond Laboratories, White Oak, Maryland, produces 400 kA at 12.5 MV with a roughly 100-nsec pulse time.) Then with an input magnetic field over a 1-cm-radius circular guide of 200 kG decreasing to 2.5 kG over a 8.1-cm-radius circular guide at the output, calculations indicate that upwards of 0.5 kA of 1-GeV protons would be achievable in such an accelerator in less than 5 m length.

Details of the eigenmode structure and calculation of these parameters are planned for future publication. The means of generation of the cyclotron eigenmode and the "loading" of ions into the potential traps will also be presented. Many considerations of stability of operation and refinement of output parameters await further theoretical investigation. In particular, the so-called "trapped particle" instability[13] between the trapped ions and the electron beam may be troublesome. One is, however, tempted to argue that because of the short lengths and short time scales for the acceleration process, plus the fact that the electrons are used just once in a single-pass fashion, most instabilities would tend to be convective and exit the machine before they have grown to such an amplitude so as to be disruptive. If such is indeed the case, and if the current estimates of maximum ion current and energy are reasonably correct the autoresonant acceleration concept offers a compact accelerator capable of pulsed output of high-energy ions at intensities orders of magnitude greater than achievable by present-day accelerators.

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Scaling Function for Critical Scattering

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The zero-field, two-point correlation function of an n-vector system in $d=4-\epsilon$ dimensions is calculated to order ϵ^2 for $T \geq T_c$. The scaling function is obtained as a closed, cutoff-independent integral. As $t=(T-T_c)/T_c \to 0$ at fixed wave vector q, the leading variation is $\hat{E}_1^{n,d}(q)t^{1-\alpha}+\hat{E}_2^{n,d}(q)t$, where α is the specific-heat exponent; thence the maximum in the scattering above T_c is located, in good agreement with high-T series-expansion estimates.

As a critical point is approached in zero field (i.e., along a line of symmetry) the correlation length diverges[1,2] as $\xi(T) \approx f_1 a t^{-\nu}$ where $t=(T-T_c)/T_c$, and a is the lattice spacing. The exponent ν depends on d, the dimensionality of the lattice, and on n, the number of "spin" components. Similarly, the scattering intensity, which is proportional to the Fourier transform, $\hat{G}(\vec{q}, T)$, of the two-point correlation function becomes large at low wave numbers $\vec{q}$.

According to the scaling hypothesis[1-4] for the critical correlations, one can write

$$\hat{G}(\vec{q}, T) \approx C t^{-\gamma} \hat{D}(q^2\xi^2), \tag{1}$$

for $t \ll 1$ and $qa \ll 1$. On adopting the normalizations $\hat{D} = -d\hat{D}/dy = 1$ at $y=0$, the constant C becomes the amplitude of the (reduced) initial susceptibility or zero-angle scattering function $\hat{\chi}_0(T)=\hat{G}(0,T)$, which diverges with exponent $\gamma=\gamma(n,d)$. Similarly, $\xi \equiv \xi_1$ must then be identified as the second-moment correlation length.[1,4] The amplitudes C and f_1 must depend on the details of the interactions, the lattice structure, etc., but the form of the scaling function $\hat{D}(x^2)$ is expected to be "universal," depending only on n and d.[5,6]

In this note we present, for the first time, an analytic calculation of the scaling function $\hat{D}(x^2)$ exact to order ϵ^2, where $d=4-\epsilon$,[7,8] for a system of general n with isotropic ($\vec{s}\cdot\vec{s}'$) interactions of finite range. Our result may be written

$$1/\hat{D}(x^2) = 1 + x^2 - 4p_n x^4 Q(x^2)\epsilon^2 + O(\epsilon^3), \tag{2}$$

where $p_n = \frac{1}{2}(n+2)/(n+8)^2$ and $Q(x^2)$ is defined by the fully convergent integral

$$Q(y) = y^{-2}\int_0^\infty dz\, [z(1+\tfrac{1}{4}z)]^{1/2} \ln[(1+\tfrac{1}{4}z)^{1/2}+\tfrac{1}{2}z^{1/2}] \times\{(1+z)^{-1} - y(1+z)^{-3} - \tfrac{1}{2}y^{-1}z^{-1}[1+y+z-(1+2y+2z+y^2-2yz+z^2)^{1/2}]\}. \tag{3}$$

The cutoff independence of this integral (see below) confirms the expected universality of $\hat{D}(x^2)$. In the limits $\epsilon \to 0$ and $n \to \infty$ the result reduces to the Ornstein-Zernike (OZ) form, $\hat{D}=(1+x^2)^{-1}$, as may be expected. (A separate exact calculation to order[9] $1/n$ has been undertaken.[10]) To leading order in ϵ the deviation from the OZ form is proportional to the exponent[7,8] $\eta=\eta(n,d)$, which determines the critical-point decay of correlation.[1,2]

In the low-momentum limit ($q \to 0$ at fixed $T > T_c$), expansion of (3) in powers of y leads to the small-x expansion $1/\hat{D}(x^2) = 1 + x^2 - \Sigma_4 x^4 + \Sigma_6 x^6 - \dots$, the form of which has been anticipated on general grounds.[1,2,4] Numerical evaluation of the definite integrals derived from (3) yields $\Sigma_{2k} = 2b_{2k}p_n\epsilon^2 + O(\epsilon^3)$, with b_4