Temporally Growing Raman Backscattering Instabilities in an Inhomogeneous Plasma

J. F. Drake and Y. C. Lee
Department of Physics, University of California, Los Angeles, California 90024
(Received 6 August 1973)

We derive the equations describing the decay of an intense, coherent electromagnetic wave into a backscattered electromagnetic wave and a plasma wave in an inhomogeneous plasma. The plasma wave and scattered electromagnetic wave are trapped in the vicinity of their cutoffs near $\omega_p = \omega_0/2$. The resulting temporally growing instabilities may explain the strong heating and the blowoff observed in recent computer simulations of Raman scattering.

We will investigate the Raman backscattering of a coherent electromagnetic (EM) wave in an inhomogeneous plasma in the cutoff region of the backscattered EM wave. This cutoff occurs when $\omega_0 \approx 2\omega_{pe}$, where ω_0 is the frequency of the incident pump wave. The WKB approximation of other authors breaks doen in this region.[1] Forsland, Kindel, and Lindman have recently carried out inhomogeneous plasma computer simulations of Raman backscattering and have found that strong electron heating and scattering occur in the plasma in the vicinity of the cutoff of the scattered EM wave.[2] More recently these authors have found that a temporally growing instability can exist in the same region.[3]

We assume that a plane-polarized EM pump wave, $E = 2\,\mathrm{Re}[E_{\omega_0}(z)\exp(-i\omega_0 t)]$, propagates into a plasma along its density gradient and is incident on an EM perturbation, E_{ω_-}, propagating in the backscatter direction. The oscillation velocity of the electrons in the electric field of one wave couples with the magnetic field of the other wave to produce a spatially dependent Lorentz force along $\hat{z}$ (the direction of the density gradient). This Lorentz force, which oscillates in time at a frequency $\omega = \omega_0 + \omega_-$, produces electron bunching enhanced by the response of the electrons to their self-consistent field, E_ω. E_ω satisfies the following equation:

$$\left[\frac{3a^2}{2\omega^2}\frac{d^2}{dz^2} - \frac{z}{L} + \frac{i\bar{\gamma}}{\omega}\right]E_\omega = -\frac{e}{m\omega_0\omega_-}\frac{d(E_{\omega_-}E_{\omega_0})}{dz}, \quad (1)$$

where $\bar{\gamma} = (\gamma + \nu_c)/2$ and γ and ν_c are the temporal growth and collisional damping rates of E_ω, respectively (ω is real). We have approximated the plasma frequency as $\omega_p^2(z) = \omega^2(1 + z/L)$, where L is the density scale length. The imaginary terms are treated as constants since we will be primarily interested in the localized region where $\omega \sim \omega_p(z)$. In the vicinity of its cutoff E_{ω_-} varies very slowly in space. The most rapid spatial variation in the plasma at this point is generated by the pump which varies approximately as $\exp(ik_0 z)$, where $k_0 \approx (\omega_0^2 - \omega^2)^{1/2}/c$ is the local wave number of the pump. If the dispersion of E_ω (of order a^2/c^2) is much smaller than $\bar{\gamma}/\omega$, it can be neglected and E_ω becomes

$$E_\omega = -(e/m\omega_-\omega_0)\epsilon_\omega^{-1} d(E_{\omega_-}E_{\omega_0})/dz, \quad (2)$$

where $\epsilon_\omega(z) = 1 - \omega_p^2(z)/\omega^2 + i\bar{\gamma}/\omega$. The neglect of the dispersion of this solution is valid if[4] $[(\omega/\bar{\gamma}k_0 L + 1)^3 - 1]k_0 L a^2/c^2 \ll 1$. The density perturbation generated by the beating of E_{ω_-} and E_{ω_0}, which can be found by inserting E_ω into Poisson's equation, will couple with the electric field of the pump to produce a current perturbation, $\Delta J_{\omega_-} = (v_{-\omega_0}/4\pi)dE_\omega/dz$, where $v_{\omega_0} = eE_{\omega_0}/m\omega_0 =$ electron oscillation velocity in E_{ω_0}. Adding ΔJ_{ω_-} to the usual wave equation for E_{ω_-}, we find

$$\left[\frac{d^2}{dz^2} + \left(\frac{\omega_-^2}{c^2}\right)\epsilon_{\omega_-} + \left(\frac{v_{\omega_0}^*}{c}\right)\frac{d}{dz}\epsilon_\omega^{-1}\frac{d}{dz}\left(\frac{v_{\omega_0}}{c}\right)\right]E_{\omega_-} = 0. \quad (3)$$

On expansion of the dielectric functions for a linear profile, Eq. (3) becomes

$$\left\{\frac{c^2}{\omega^2}\frac{d^2}{dz^2} + \Delta - \left(\frac{z}{L} + \frac{i\bar{\gamma}}{\omega}\right) + \frac{3v_0^2/c^2}{z/L - i\bar{\gamma}/\omega}\left[1 + \frac{i/k_0 L}{z/L - i\bar{\gamma}/\omega}\right]\right\}E_{\omega_-} = 0, \quad (4)$$

where $\Delta = \omega_-^2/\omega^2 - 1$ represents the mismatch between the cutoffs of E_ω and E_{ω_-} in the absence of the pump. We have assumed that $k_0^{-1}|d\ln E_{\omega_-}/dz| \ll 1$. The last term in Eq. (4) arises from electron bunching produced by the density gradient and significantly modifies the spatial amplification of E_{ω_-} if $\omega/\bar{\gamma}k_0 L \gtrsim 1$. If $\omega/\bar{\gamma}k_0 L \ll 1$,

E_{ω_-} is spatially amplified in the region $z^2/L^2 < 3v_0^2/c^2 - \bar{\gamma}^2/\omega^2 \ll 1$. We will investigate the temporally growing solutions to Eq. (4) which can exist if E_{ω_-} is trapped by the plasma. After a change of variables, Eq. (4) becomes

$$[d^2/dx^2 + \kappa^2(x)]E_{\omega_-}(x) = 0, \qquad (5)$$

where

$$\kappa^2(x) = \alpha^2 + \frac{1-(x/\delta)^2}{x - i\epsilon} + \frac{i\beta}{(x-i\epsilon)^2},$$

$$x = (k_0 z)(v_0/c)^{1/2}N\mathcal{E},$$

$$\alpha^2 = [(\omega_-^2/\omega^2) - 1](c/v_0)N^2/3\mathcal{E}^2,$$

$$\epsilon = (\bar{\gamma}/\omega)(c/v_0)\mathcal{E}N^2, \quad \delta = \sqrt{3}\,\mathcal{E}^{3/2}N^2,$$

$$\mathcal{E} = 1 - (\bar{\gamma}/\omega)^2(c/v_0)^2/3, \quad \beta = N(v_0/c)^{1/2},$$

$$N = (k_0 L)(v_0/c)^{3/2}.$$

If $\beta/\epsilon \ll 1$, $\kappa^2(x)$ has the form shown in Figs. 1 and 2 when $\delta/\epsilon = [3(v_0/c)^2 - (\bar{\gamma}/\omega)^2]^{1/2}/(\bar{\gamma}/\omega)$ is greater and less than one, respectively.

We expect E_{ω_-} to grow temporally at nearly the homogeneous rate ($\bar{\gamma}/\omega = \sqrt{3}\,v_0/c$) when the density gradient is very weak so $\delta^2/\epsilon^2 \ll 1$. In the region where E_ω is resonant ($|x| < \epsilon$), Eq. (5) can then be approximated as

$$\{d^2/dx^2 + \alpha^2 + (i/\epsilon)[1-(x/\delta)^2]\}E_{\omega_-} = 0. \qquad (6)$$

Equation (6) is of the same form as the quantum-mechanical harmonic-oscillator equation with a complex potential. In order that the solution to Eq. (6) be spatially decreasing at large $|x|$, we require $(\alpha^2 + i/\epsilon)\delta\sqrt{\epsilon} = 2(n+\frac{1}{2})\exp(i\pi/4)$, where n is a non-negative integer. Inserting the definitions of δ, ϵ, and α, we solve for the growth rate

$$\frac{\gamma}{\omega} = -\frac{\nu_c}{\omega} + 2\sqrt{3}\left(\frac{v_0}{c}\right)\left[1 - \frac{n+\frac{1}{2}}{N(2\sqrt{3})^{1/2}}\right]. \qquad (7)$$

The requirement that $\delta^2/\epsilon^2 \ll 1$ implies that

$$N = k_0 L(v_0/c)^{3/2} \gg n + \tfrac{1}{2}. \qquad (8)$$

Under this restriction, $\beta/\epsilon \ll 1$ and the wave is confined to the region $|x| \ll \epsilon$, justifying our previous assumptions. The frequency mismatch, $\omega_-^2 - \omega^2$ (represented by α^2), can only assume discrete values so that the trapped EM wave does not undergo phase mixing. Note also that $\alpha^2 > 0$. We call this instability the "resonant" instability. This instability will not produce a large backscattered wave since the energy is primarily confined to the resonant region of the electrostatic wave. The growth rate in Eq. (7) converges to the homogeneous value as $k_0 L$ becomes very large.

We now consider a somewhat stronger density gradient (still weak enough to neglect β). We expect unstable waves to have growth rates much smaller than the homogeneous value so $\delta^2/\epsilon^2 \gg 1$. In this case $\kappa^2(x)$ has a simple pole at the cutoff of E_ω (at $x = 0$) bounded on either side by cutoffs ($\kappa^2 = 0$). A well is therefore formed which can trap the backscattered waves. The well is produced by ΔJ_{ω_-}, which is antiparallel to the usual linear currents generated by E_{ω_-}. ΔJ_{ω_-} can locally reverse the sign of the total current generated by E_{ω_-} near the cutoff of E_{ω_-}, where the total linear currents are small. The sign of $\kappa^2(x)$ is therefore locally reversed, and the well is formed as shown in Fig. 2.

For convenience, we define $\tilde{E}(x,t)$ to be a solu-

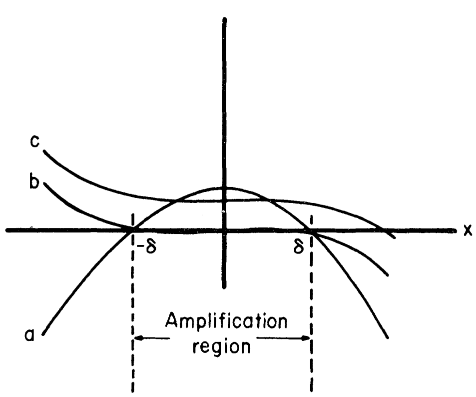

FIG. 1. $\delta^2/\epsilon^2 \ll 1$ (weak inhomogeneity). We plot (a) $\mathrm{Im}\kappa^2(x)$; (b) $\mathrm{Re}\kappa^2(x)$ with $\alpha = 0$; and (c) $\mathrm{Re}\kappa^2(x)$ with $\alpha^2 > 0$.

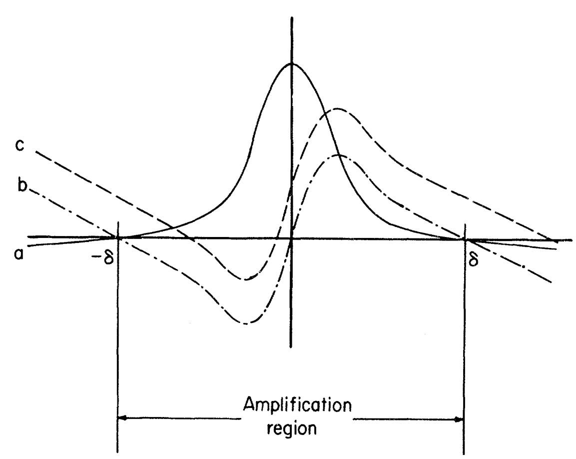

FIG. 2. $\delta^2/\epsilon^2 \gg 1$ (strong inhomogeneity). We plot (a) $\mathrm{Im}\kappa^2(x)$; (b) $\mathrm{Re}\kappa^2(x)$ with $\alpha = 0$; and (c) $\mathrm{Re}\kappa^2(x)$ with $\alpha^2 > 0$.

tion to Eq. (5) which contains the oscillating time dependence of E_{ω_-}, but does not contain the temporal growth. $\tilde{E}$ represents a wave which propagates and is trapped in the well. Since the magnitude of $\tilde{E}$ is independent of time, in each circuit of the well, $\tilde{E}$ can have no net amplification. The energy absorbed by $\tilde{E}$ from the pump must be balanced by the energy lost through the evanescent region. The spatial phase variation of $\tilde{E}$ must be adjusted so that phase mixing of the wave trapped in the well does not take place. Mathematically stated, we must obtain solutions to Eq. (5) which correspond to outgoing waves at large negative x and which are spatially decreasing at large positive x. As in the previous case, Eq. (5) is an eigenvalue equation with only certain values of α permitted where ϵ is fixed for a given α.

We first consider the case when the length of the evanescent region is greater than or of the order of the length of the well ($\alpha^2 < 0$ or $\delta^2\alpha^4 \lesssim 1$). The well is primarily amplifying in this case. $\tilde{E}$ can not have zero net amplification in one circuit of the well since the large barrier will not allow enough energy to pass to balance the energy gain in the amplifying well. Exponentially unstable solutions therefore do not exist.

If the evanescent region is very short and the well is very wide ($\delta^2\alpha^4 \gg 1$), Eq. (5) assumes the form of Budden's equation with an energy source in the vicinity of the singularity and short evanescent region. We match the solution in this region with the turning-point solution near the right-hand side of the well using a transformation first used by Langer and later applied by Baños[6] in plasma physics. We obtain the following dispersion relation[7]:

$$\tfrac{2}{3}\alpha^3\delta^2 - i\epsilon\alpha + \tfrac{1}{2}i\ \ln[\exp(\pi/\alpha) - 1] = (n + \tfrac{1}{4})\pi, \qquad (9)$$

where n is a positive integer. The first term on the left-hand side of the above equation, which represents the integrated phase of E_{ω_-} across the well, must equal $(n+\frac{1}{4})\pi$ to prevent phase cancelation of the trapped waves. The spatial amplification of $\tilde{E}$ as it scatters off the singularity at the left-hand side of the well (logarithmic term) must balance the spatial damping of $\tilde{E}$ occurring primarily in the right-hand side of the well. Inserting the definitions for α, ϵ, and δ, we solve for the growth rate

$$\frac{\gamma}{\omega} = -\frac{\nu_c}{\omega} + \left(\frac{2\nu_0}{c}\right)\frac{\ln\{\exp[2\pi^2N^4/(n+\frac{1}{4})]^{1/3} - 1\}}{[4\pi N^2(n+\frac{1}{4})]^{1/3}}, \qquad (10)$$

where $N = k_0L(v_0/c)^{3/2}$. The assumptions that $\delta^2/\epsilon^2 \gg 1$, $\alpha^4\delta^2 \gg 1$, and $\beta/\epsilon \ll 1$ require

$$(v_0/c)^{1/2} \ll N = k_0L(v_0/c)^{3/2} \ll (n+\tfrac{1}{4})\pi. \qquad (11)$$

When the pump is far above the homogeneous threshold, we can approximate the inhomogeneous threshold for this instability by demanding that the second term on the right-hand side of Eq. (10) be positive. We denote this instability the "nonresonant" instability since the well extends beyond the resonant region of E_ω. The backscattered EM waves in this instability grow exponentially in time only when they are weakly trapped in the well (the evanescent region is short). We note that in this instability the plasma with the pump as the energy source traps, amplifies, and emits EM ratiation very much like a laser.

Comparing the inequalities in Eqs. (8) and (11), as the density gradient becomes too strong for a given resonant mode to exist, its nonresonant counterpart will be unstable. In spite of the apparently different trapping mechanisms which produce the resonant and nonresonant instabilities, the similarities between the inequalities in Eqs. (8) and (11) seem to suggest that they are different limits of the same instability. Combining the inhomogeneous threshold of the nonresonant instability with the homogeneous threshold of the resonant instability, we arrive at an approximate minimum threshold for instability:

$$v_0/c = \tfrac{1}{2}(E_{\omega_0}{}^2/4\pi n_0mc^2)^{1/2} = 3^{-1/2}(\nu_c/\omega) + 0.52(k_0L)^{-2/3}. \qquad (12)$$

We will now speculate on probable saturation mechanisms of the instabilities discussed above. The trapped waves will grow rapidly in time and begin to deplete the pump. Since the unstable waves are eigenmodes of a well whose shape is a function of the pump intensity, a small depletion of the pump may result in a shift in the frequency of the unstable eigenmodes of the well. Therefore, the initially unstable waves saturate although waves with slightly different frequencies will then be unstable. The EM pressure of the scattered wave, which is strongly peaked in the vicinity of the well, may eventually expel the electrons and ions and therefore blow off the entire plasma which is towards the direction of incidence from the well. This blowoff has been observed in Ref. 2.

The plasma will be strongly heated in the vicinity of the cutoff of the backscattered EM wave as the instabilities evolve. Since the frequencies of the plasma wave and backscattered EM wave are approximately equal, half the energy leaving the pump will go directly into the unstable plasma wave. All of the energy in this electrostatic wave will eventually be locally absorbed by the plasma near the cutoff of the EM wave since the plasma wave is nonpropagating. When the density gradient is weak, most of the energy in the scattered EM wave is trapped in the resonant region of the plasma wave, and therefore a large fraction of this energy will eventually be locally absorbed by the plasma. In stronger density gradients most of the energy in the scattered EM wave will escape from the plasma.

The authors would like to acknowledge helpful conversations with A. Baños, Jr., B. D. Fried, J. M. Kindel, and G. Schmidt.

*Work partially supported by the U.S. Office of Naval Research under Grant No. N00014-69-A-0200-4023; the National Science Foundation under Grant No. GP-22817; and the U.S. Atomic Energy Commission under Contract No. AT(04-3)-34 P.A. 157.

[1]M. V. Rosenbluth, Phys. Rev. Lett. 29, 565 (1972); C. S. Liu, Bull. Amer. Phys. Soc. 17, 1065 (1972); D. Pesme, G. Laval, and R. Pellat, Phys. Rev. Lett. 31, 203 (1973).

[2]D. W. Forslund, J. M. Kindel, and E. L. Lindman, Phys. Rev. Lett. 30, 739 (1973).

[3]J. M. Kindel, private communication.

[4]The inversion of Eq. (1) has been carried out by J. Drake and Y. C. Lee [UCLA Plasma Physics Report No. PPG 156 (unpublished)] by obtaining an exact integral representation for E_ω and then expanding the integral.

[5]Equation (4) has been generalized to the case of two-dimensional scattering ($\vec{E}_{\omega_0} \cdot \nabla = 0$) by Drake and Lee, Ref. 4.

[6]A. Baños, Jr., UCLA Plasma Physics Report No. PPG 124 (to be published).

[7]Drake and Lee, Ref. 4.

Study of the Bend Elastic Constant near a Smectic-A–Nematic Phase Transition

P. E. Cladis
Bell Laboratories, Murray Hill, New Jersey 07974

(Received 16 April 1973; revised manuscript received 27 September 1973)

The temperature dependence of the bend elastic constant of N-*p*-cyanobenzilidene-*p*-*n*-octyloxyaniline near the smectic-A–nematic transition temperature T_c is found to obey the power law $(T - T_c^*)^{-\gamma_3}$ with classical values for γ_3 in purer samples. Deviations from this value are discussed. Scanning calorimetry indicates that for purer samples the transition is either weakly first order or a λ transition.

Recently, de Gennes[1] has formulated an analogy between the superconducting-normal metal transition and the second-order nematic–smectic-A transition. In the former case, the fluctuations of temporal Cooper pairs in the normal state lead to an increase in the normal-state diamagnetic susceptibility near T_c.[2] In the latter, the density fluctuations in the nematic phase (which may be described by an order parameter, ψ), due to the continual production of evanescent, submicroscopic smectic regions, lead to an increase in the nematic elastic constants[3] of bend (K_3) and twist (K_2) so that

$$\delta K_i = K_i(T) - (K_i)_0 = (\text{const})_i (T - T_c)^{-\gamma_i}, \qquad (1)$$

where T_c is the nematic-smectic transition temperature, $(K_i)_0$ is the value without the enhancement due to the order-parameter fluctuations, $i = 2, 3$. Here measurements are presented on the temperature dependence of $K_3(T)$ for CBOOA (N-*p*-cyanobenzilidene-*p*-*n*-octyloxyaniline). Fitting the data with the functional form of Eq. (1), one determines the four parameters $(K_i)_0$, $(\text{const})_i$, γ_i, and T_c which best fit the data. This is done by the well-known technique of minimizing the weighted sum of the squares of the difference between the measured and computed values (i.e., χ^2) for $K_i(T)$ with respect to the four parameters. In the case of a second-order transition, T_c may be measured independently and so does not enter as a parameter in the fit. In the case of a first-order transition the T_c in Eq. (1) should be replaced by $T_c^* < T_{c,\text{meas}}$ which is to be found.

CBOOA was chosen because McMillan[4] recently indicated that he was unable to observe a latent heat associated with the smectic-A–nematic transition and that the transition was presumably second order. Figure 1 shows a differential thermal analysis of the transition made on 9.1 mg of purified CBOOA at constant pressure. An apparent