ma interaction, our simulation parameters scale as follows. For a CO_2 laser, the power is 3×10^{14} W/cm^2, the density scale length is $L=2\times10^{-2}$ cm, the time duration of the simulation is 6 psec, the interaction is at a density of one seventh the critical density, and the electron temperature is 1 keV.

On the other hand, for a neodymium-glass laser the power is 3×10^{16} W/cm^2, the density scale length is 2×10^{-3} cm, the time duration of the simulation is 0.6 psec, the interaction is at one seventh the critical density, and the temperature is still 1 keV.

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Temporal Evolution of a Three-Wave Parametric Instability*

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We solve exactly the temporal evolution and spatial dependence of a three-wave parametric instability in an inhomogeneous plasma. An initial fluctuation develops into a pulse and grows initially with the same growth rate as it would in a homogeneous plasma. Growth continues until convection saturation occurs. The pulse broadens as it grows and eventually assumes the form of a totally amplified region flanked by two shock fronts. Effects of damping are also discussed.

The basic equations governing the growth of a parametric three-wave instability in an inhomogeneous medium, as discussed by Rosenbluth and Nishikawa,[1] are

$$\nu_1 a_1 + \partial a_1/\partial t + v_1\,\partial a_1/\partial x = \gamma_0 a_2 e^{i\kappa' x^2/2}, \quad \nu_2 a_2 + \partial a_2/\partial t - v_2\,\partial a_2/\partial x = \gamma_0 a_1 e^{-i\kappa' x^2/2}, \tag{1}$$

where γ_0 is the growth rate for the instability in a homogeneous medium and $\kappa' = (d/dx)[k_0(x)+k_1(x)+k_2(x)]$, assumed to be a constant, determines the phase mismatch between the decay waves. In deriving (1) the amplitudes of the waves propagating (oppositely directed) in the plasma have been written $A_i = a_i(x,t)\exp(i\omega_i t - ik_i x - i\int_0^x \Delta k_i\,dx')$, and it is assumed that $a_i(x,t)$ varies slowly in space compared to $\exp(-ik_i x)$. This assumption is clearly inconsistent with (1) for $\kappa'|x|>k_{ix}$, and thus this equation is limited to this range of x. In the following analysis we will assume $|x|\ll k_{ix}/\kappa'$, and also less than the characteristic size of the system. Taking a Laplace transform in time and letting $a_i = \hat{a}_i e^{-i\kappa' x^2/4}$, we find

$$(p+\nu_1)\hat{a}_1 + \tfrac{1}{2}i\kappa' v_1 x\hat{a}_1 + v_1\,\partial\hat{a}_1/\partial x = \gamma_0\hat{a}_2 + \hat{a}_1(0), \quad (p+\nu_2)\hat{a}_2 + \tfrac{1}{2}i\kappa' v_2 x\hat{a}_2 - v_2\,\partial\hat{a}_2/\partial x = \gamma_0\hat{a}_1 + \hat{a}_2(0). \tag{2}$$

We choose initial values as $\hat{a}_1(0)=0$, $\hat{a}_2(0) = (-\bar{a}/\sqrt{\kappa'})\delta(x-x_0)$, where $\bar{a}$ represents the thermal level of the wave amplitude, normalization chosen such that $\int a_2\,dx = \bar{a}/\sqrt{\kappa'}$, noting that $(\kappa')^{-1/2}$ is the region of wave interaction. Substituting $a=\hat{a}_1 \exp\{[(p+\nu_1)/v_1-(p+\nu_2)/v_2](x-x_0)/2\}$ and eliminating $\hat{a}_2$ we find

$$\frac{\partial^2 a}{\partial x^2}+\frac{1}{4}\left[\kappa' x-i\left(\frac{p+\nu_1}{v_1}+\frac{p+\nu_2}{v_2}\right)\right]^2 a+\left(\frac{\gamma_0{}^2}{v_1 v_2}+\frac{i\kappa'}{2}\right)a=\frac{\gamma_0\bar{a}\delta(x-x_0)}{\sqrt{\kappa'}v_1 v_2}. \tag{3}$$

Substituting

$$x=x'/\sqrt{\kappa'}+(i/\kappa')[(p+\nu_1)/v_1+(p+\nu_2)/v_2],$$

we find

$$\partial^2 a/\partial x'^2+\tfrac{1}{4}x'^2 a+(\lambda+\tfrac{1}{2}i)a=(\bar{a}\lambda/\gamma_0)\delta(x'-x_0'), \tag{4}$$

with $\lambda=\gamma_0{}^2/\kappa' v_1 v_2$. The solution to Eq. (4), well behaved at $\pm\infty$ and satisfying the jump conditions at $x=x_0$, is

$$a(x,p)=(\lambda\bar{a}/\gamma_0)[\psi_{\mathrm{I}}(x)\psi_{\mathrm{II}}(x_0)\theta(x-x_0)+\psi_{\mathrm{II}}(x)\psi_{\mathrm{I}}(x_0)\theta(x_0-x](\psi_{\mathrm{I}}'\psi_{\mathrm{II}}-\psi_{\mathrm{II}}'\psi_{\mathrm{I}})^{-1}, \tag{5}$$

where

$$\psi_{\mathrm{I}}(x)=D_{-i\lambda}x'e^{i\pi/4}, \quad \psi_{\mathrm{II}}(x)=D_{i\lambda-1}(-x'e^{-i\pi/4})$$

are parabolic cylinder functions defined by

$$D_\nu(z)=[e^{-z^2/4}/\Gamma(-\nu)]\int_0^\infty e^{-zt-t^2/2}t^{-\nu-1}dt. \tag{6}$$

Substituting (for $x>x_0$) the integral expression (6) into Eq. (5), we find

$$a(x,p)=(\lambda\bar{a}/\gamma_0)\pi^{-1}\sinh(\pi\lambda)\exp(-\tfrac{1}{4}x'^2+\tfrac{1}{4}ix_0'^2)$$
$$\times\int_0^\infty ds\int_0^\infty dt\,\exp(-e^{i\pi/4}sx'+e^{-i\pi/4}tx_0'-\tfrac{1}{2}s^2-\tfrac{1}{2}t^2)\,s^{i\lambda-1}t^{-i\lambda}. \tag{7}$$

Changing integration variables through the substitutions $s=e^{i\pi/4}s'$, $t=e^{-i\pi/4}s''$, and inverting the Laplace transform, the integration over p gives rise to

$$\delta(t\sqrt{\kappa'}v_1 v_2/(v_1+v_2)-s'-s''-(x-x_0)v_2\sqrt{\kappa'}/(v_1+v_2)),$$

allowing the integration over s'' to be performed trivially. Substituting $s'=z[t+v_1{}^{-1}(x_0-x)]v_1v_2\sqrt{\kappa'}/(v_1+v_2)$ and introducing the variables $X=\sqrt{\kappa'}x$, $T=v_1\sqrt{\kappa'}t$, $\alpha=v_2/v_1$, and $V=\sqrt{\kappa''}v_1$, we find

$$a_1(X,T)=\frac{(v_1v_2)^{1/2}}{v_1+v_2}\frac{\sinh(\pi\lambda)}{\pi}\sqrt{\lambda}\,\bar{a}\exp\left(\frac{-i(X+X_0)\alpha(T+X_0-X)}{2(1+\alpha)}\right)K_1K_2\int_0^1\frac{dz}{z}\left(\frac{z}{1-z}\right)^{i\lambda}\exp\left[\frac{-i\lambda}{Q}(z-\tfrac{1}{2})\right], \tag{8}$$

$$K_1\equiv\exp\left(-\frac{\nu_1}{V}\frac{\alpha T+X+X_0}{1+\alpha}\right),\quad K_2\equiv\exp\left(-\frac{\nu_2}{V}\frac{T-X-X_0}{1+\alpha}\right),\quad Q\equiv\lambda(1+\alpha)^2\frac{\alpha(T-X+X_0)}{\alpha T+X-X_0}.$$

For $X<X_0$, interchange X and X_0 and subscripts 1 and 2. The casual region originating at $X=X_0$, $T=0$ is given by $Q>0$. The integrand in (8) possesses saddle points at $z=\frac{1}{2}\pm(\frac{1}{4}-Q)^{1/2}$. If $Q>\frac{1}{4}$, the appropriate saddle is the one located in the upper-half z plane, giving ($\pi\lambda\gg1$)

$$a_1(X,T)=\frac{(v_1v_2)^{1/2}}{v_1+v_2}\frac{\bar{a}2^{-3/2}}{\sqrt{\pi}}\left[\frac{1}{2\sqrt{Q}}\left(1-\frac{1}{4Q}\right)\right]^{-1/2}\exp\left(\frac{-i(X+X_0)\alpha(T+X_0-X)}{2(1+\alpha)}\right)K_1K_2$$
$$\times\exp\left[\frac{\lambda}{\sqrt{Q}}\left(1-\frac{1}{4Q}\right)^{1/2}+2\lambda\sin^{-1}\left(\frac{1}{2\sqrt{Q}}\right)\right]. \tag{9}$$

If $Q<\frac{1}{4}$, the saddle points are located on the real axis, the one closer to $z=0$ giving the dominant contribution. We then have

$$a_1(X,T)=[(v_1v_2)^{1/2}\bar{a}/(v_1+v_2)\sqrt{2\pi}]\exp\{i(X+X_0)[X_0-X-\alpha T-\alpha(X_0-X+T)]/4(1+\alpha)\}$$
$$\times K_1K_2e^{\pi\lambda}e^{i\lambda/2Q}. \tag{10}$$

The condition $Q\ll\frac{1}{4}$ corresponds, for any point in space, to large T, and thus Eq. (10) represents the limiting value of the amplitude. When $Q\sim\frac{1}{4}$ the amplitude is in the process of saturating, and the saddle-point evaluation is invalid.

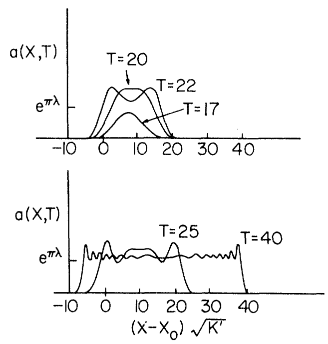

FIG. 1. Temporal evolution of a parametric instability, neglecting damping. A δ-function initial value is given at $x=0$. In this case $v_1=5v_2$, so the instability takes the form of a broadening pulse moving to the right. The initial growth at the point $x-x_0=(v_1-v_2)t/2$ is given by γ_0. $\lambda=4$.

For early times, but within the causal forward cone $Q>0$, $Q\sim\lambda(1+\alpha)\{\alpha\Delta T[|X-X_0|+\alpha\Delta T/(1+\alpha)]\}^{-1}$; thus the initial growth at any x is given by

$$a_1(X,\Delta T)\sim\exp\left[2\left(\frac{\lambda\,\alpha[|X-X_0|+2\Delta T/(1+\alpha)]\Delta T}{1+\alpha}\right)^{1/2}\right]. \tag{11}$$

In particular, at $X=X_0$ the initial growth is exponential with $\gamma_{\rm eff}=\gamma_0^2(v_1v_2)^{1/2}/(v_1+v_2)$. We first discuss (9) and (10) in the absence of damping, $\nu_1=\nu_2=0$. At any fixed T the maximum amplitude occurs at $X=X_0+T(1-\alpha)/2$, where Q is a minimum. Thus, the amplitude takes the form of a single pulse convecting in the direction of the fastest moving wave with a position[2] $x=x_0+(v_1-v_2)t/2$. At this point $Q=4\lambda/\alpha T^2$, and thus the amplitude initially grows exponentially with a growth rate of γ_0. The relevant growth rate for the instability is γ_0, unless the convecting mode is limited either by the finite extent of the plasma or by the region of effective phase matching, $|x|<k_{ix}/\kappa'$. The width Δ of this peak is given approximately by $\Delta=(v_1+v_2)t/2$, the pulse thus broadening greatly as it moves. The convection saturation of the amplitude $a_1(x,t)\sim\exp(\pi\gamma_0/\kappa'v_1v_2)$ as $Q\to\frac{1}{4}$ is physically due to the dominance in the spectrum of $a_1(x,t)$ of wavelengths longer than the interaction length $(\kappa')^{-1/2}$. In Fig. 1 is shown an example of the temporal evolution of such a mode, calculated by numerical integration of Eq. (8). Note that the amplitude temporarily overshoots its final value by about 50%. In Fig. 2 is shown the value of the amplitude at the center of the pulse as a function of time.

The inclusion of damping can produce qualitative changes in the temporal evolution of the instability. In particular, if the amplitude maximum is convecting rapidly ($v_1\gg v_2$) there may not be significant amplification at $x=x_0$ at any time, but the amplification at the pulse maximum can still be quite large. In Fig. 3 is shown a convective mode with $v_1=2v_2$. The left-moving wave is damped, $\nu_2/\sqrt{\kappa'}v_2=0.2$, and the right-moving wave undamped.

Consider for any X the maximum of $a_1(X,T)$ over all T. As long as $\nu_1,\nu_2<\gamma_0$ this occurs at a time T given approximately by $4Q=1$. Substituting into Eq. (10) we find the envelope

$$a_{\rm max}(x)\sim\sinh(\pi\lambda)\exp\{-(\kappa')^{-1/2}(\nu_1/2v_1+\nu_2/2v_2)[(X-X_0)^2+16\lambda]^{1/2}-(\kappa')^{-1/2}(\nu_1/2v_1-\nu_2/2v_2)(X-X_0)\}. \tag{12}$$

In particular, examining the point $x=x_0$ we find ($\pi\lambda\gg1$)

$$a_{\rm max}(X_0)\sim\exp\left\{\frac{2\gamma_0^2}{\kappa'\nu_1\nu_2}\left[\frac{\pi}{2}-\frac{(v_1v_2)^{1/2}}{\gamma_0}\left(\frac{\nu_1}{v_1}+\frac{\nu_2}{v_2}\right)\right]\right\}, \tag{13}$$

and thus the condition for absolute instability,[2] $\gamma_0>(v_1v_2)^{1/2}(\nu_1/v_1+\nu_2/v_2)$, determines whether amplification will occur at the same point in space at which a fluctuation occurs. In Fig. 4 $a_{\rm max}(x)$ is shown for various values of the collision frequency.

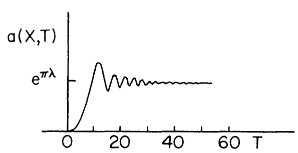

FIG. 2. Amplitude of the instability at the center of the pulse, $x-x_0=(v_1-v_2)t/2$, for the case of Fig. 1, as a function of time.

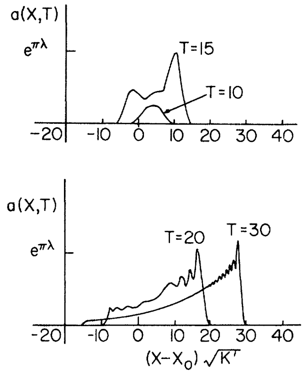

FIG. 3. Temporal evolution of a parmaetric instability, including damping. $v_1 = 2v_2$, $\lambda = 4$, $\nu_1 = 0$, $\nu_2/\sqrt{\kappa'}v_2 = 0.2$.

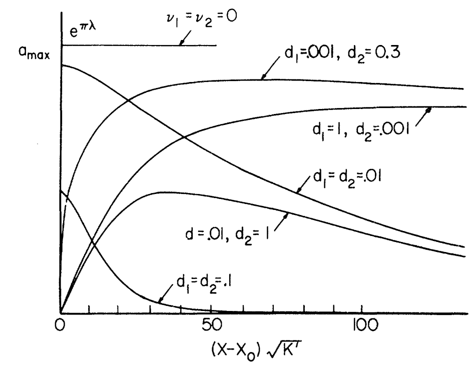

FIG. 4. Maximum (over t) amplitude $a_{max}(x)$ for various collision frequencies, $d_i = \nu_i/\sqrt{\kappa'}v_i$. In this case $\lambda = 4$, so the threshold for absolute instability occurs when $d_1 + d_2 = \pi$.

We have noted that the WKB approximation is valid for $\kappa' x < k$, or roughly $x < L$, with L a plasma scale height. We note from (12) that in the case of interest $(\nu_2/v_2 \gg \nu_1/v_1)$ the amplitude is an increasing function of x, and moreover the largest allowable value of $(X - X_0)_{max} = (kL)^{1/2} \gg 1$. As we are interested in a_{max} for the case $\lambda \approx 1$, we expand (12) for this value to find that the peak amplification anywhere in the plasma is $\exp[\lambda(\pi - 4\nu_2/kv_2)]$. Thus, only in the case of very heavy damping will the amplification be affected.

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Dilation and Conformal Covariance of Multipoint Correlation Functions and Dimensions of Fluctuating Quantities at the Critical Point of Fluids

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The operator algebra and an identity in the statistical mechanics of fluids are used to demonstrate that the cumulant average of arbitrary products of appropriate combinations of local fluctuating energy and density are scale and conformal covariant at the critical point. These combinations are uncorrelated and are found by diagonalizing a matrix whose elements are energy and density derivatives of pressure-density and pressure-energy correlations. The eigenvalues are the dimensions of the appropriate variables.

In this Letter we apply two lines of thought (which we have discussed elsewhere[1,2]) to the simultaneous fluctuations of local density, $\delta\rho(\vec{x})$, and local energy density, $\delta e(\vec{x})$, in a fluid at its critical point. We identify two local fluctuating quantities, each of which is a linear combination