Analysis of Three-Hadron Final States

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Unitarity implies a rapid variation (singularity) in an amplitude usually assumed to be slowly varying in simple pairwise final-state–interaction fits to three-body final states. Neglect of this singularity casts doubts on recent analyses of $\pi+N\rightarrow\pi+\pi+N$, among others. Tentative suggestions for including the singular term are offered.

Final states with three or more hadrons are an important source of information on the interaction of unstable particles both in nuclear and particle physics. Unfortunately, our theoretical understanding of such states has provided very little basis for the various phenomenological schemes used to analyze them. In this paper we show that the general constraints of unitarity lead to new conditions on the multibody amplitudes that have been ignored in analyses up to now. Their neglect may account for a wide range of difficulties ranging from the "wrong" sign for $\pi+N\rightarrow\pi+\Delta$ amplitudes[1] obtained from analysis[2] of the reaction $\pi+N\rightarrow\pi+\pi+N$ to "wrong" values of the neutron-neutron scattering length obtained from the reaction $n+d\rightarrow n+n+p$.[3] We were alerted to this difficulty by the results of numerical calculations we have been doing in a unitary dynamical theory of the π-N system[4,5] and in a model related to the n-d problem.[6]

Since the condition we will obtain applies to both the relativistic and nonrelativistic problem, we will present its derivation in a symbolic manner that applies to both. We shall only derive the result for three-body final states, but much of the analysis is appropriate to more particles. Consider an amplitude $T_{2,3}$ for going from a two-body state to a three-body state in a particular channel (i.e., fixed J, L, T, etc.) (our results would equally well apply to a one-to-three decay amplitude). Decompose $T_{2,3}$ into a sum of terms depending on which pair interacts last,

$$T_{2,3}=\sum_i f_i \tau_i, \tag{1}$$

where τ_i is the two-body scattering amplitude of the j-kth pair ($i\neq j\neq k$) and f_i is its coefficient in $T_{2,3}$. This is the standard isobar or Faddeev or multiple-scattering decomposition. Usually τ_i is taken to be dominated by a particular partial wave l_i, and then a factor of $q_i^{-l_i}$ (where q_i is the j-k relative momentum) is explicitly removed from f_i. Apart from this and other similar threshold kinematic factors, f_i is usually assumed to be slowly varying for a particular incident two-body partial wave and energy and hence approximated by a constant over the final-state phase space. We shall now use unitarity to show that in fact f_i has a strong dependence on q_i.

The unitarity relation for $T_{2,3}$ is

$$\operatorname{Im}T_{2,3}=T_{2,2}\rho_2T_{2,3}^*+T_{2,3}\rho_3T_{3,3}^{c*}+\sum_i T_{2,3}\rho_{2,i}\tau_i^*, \tag{2}$$

where $T_{2,2}$ is the elastic scattering amplitude; ρ_2,ρ_3 are the two- and three-body phase space; and $\rho_{2,i}$ is the two-body phase space of the ith pair. $T_{3,3}^{c}$ is the connected three-to-three amplitude. Equation (2) is represented diagrammatically in Fig. 1. From (1) and the definition of the imaginary part we also have[7]

$$\operatorname{Im}T_{2,3}=\sum_i f_i \operatorname{Im}\tau_i+\sum_i(\operatorname{Im}f_i)\tau_i^*. \tag{3}$$

We are not strictly interested in $\operatorname{Im}f_i$, but only in its singular or absorptive part (abbreviated Abs) due to singularities in the pair subenergies. Singularities in the total energy relate to the total energy dependence of the f_i that does not concern us here. The first two terms in (2), since they involve cutting all connected particle lines, lead to just such total energy singularities.[8] Hence we drop them and write [combining (3) + (2) and replacing Im by Abs]

$$\sum_i f_i \operatorname{Im}\tau_i+\sum_i (\operatorname{Abs}f_i)\tau_i^*=\sum_i T_{2,3}q_i\tau_i^*, \tag{4}$$

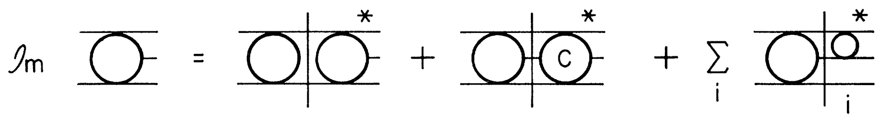

FIG. 1. Diagrammatic representation of the unitary relation, Eq. (2).

where we have used the fact that $q_i=\rho_{2i}$. Putting (1) in (4) we obtain

$$\sum_i f_i \,\mathrm{Im}\tau_i + \sum_i (\mathrm{Abs} f_i)\tau_i^* = \sum_i f_i \tau_i q_i \tau_i^* + \sum_{j\neq i} f_j \tau_j q_i \tau_i^*. \tag{5}$$

The first terms on the left- and right-hand sides of (5) cancel by two-body unitarity ($\mathrm{Im}\tau_i=\tau_i q_i \tau_i^*$), and we are left with

$$\sum_i (\mathrm{Abs} f_i)\tau_i^* = \sum_{j\neq i} f_j \tau_j q_i \tau_i^*. \tag{6}$$

Equating coefficients of τ_i^* we get

$$\mathrm{Abs} f_i = q_i \sum_{j\neq i} f_j \tau_j. \tag{7}$$

This is our major result. f_i has a singular part proportional to q_i times

$$\sum_{j\neq i} f_j \tau_j,$$

which is just the "non-i" part of $T_{2,3}$. Hence the strength of the singular part of f_i depends on the amount of $T_{2,3}$ in the other isobar channels. Equation (7) is somewhat symbolic since the arguments of f_j and τ_j are not written in. τ_j is a function of q_j. At the i-channel threshold ($q_i=0$), q_j is uniquely determined from energy and momentum conservation, but for arbitrary q_i there is a range of q_j that must be integrated over in (7). Hence (7) represents a set of coupled integral conditions on the f_i's. The full content of these conditions is under investigation. In this paper we only call attention to its general features and give some simple approximate prescriptions for implementing them.

Essentially this same result (7) was obtained by one of us in 1967 in the context of the Faddeev equations,[9,10] but its importance to the analysis of three-body final states was not at all appreciated. The 1967 paper shows the relation of (7) to Watson's theorem and how it gives the entire amplitude (not just a part of it) the phase δ. The result (7) may also be easily obtained by a study of the singularities of the multiple-scattering expansion.[6,8] One sees that the singularity comes from the last rescattering, and hence its form can be correctly obtained, for example, from perturbation theory, even if the perturbation expansion diverges.[6]

We now turn to the question of how (7) might be included in an analysis. We have yet to explore fully the consequences of (7) and thus our remarks are preliminary. If we rewrite (1) in terms of the more usual isobar parametrization[2] (we assume τ_j acts only in the l_j wave),

$$T_{2,3}=\sum_j f_j \tau_j^{(l_j)} q_j^{-l_j}, \tag{1'}$$

Eq. (7) implies

$$f_j = A_j + i q_j^{l_j+1} B_j = A_j + i q_j^{l_j+1} \sum_{k\neq j} f_k \tau_k^{(l_k)} q_k^{-l_k}, \tag{7'}$$

where A_j and B_j are complex functions with no physical q_j singularities. A number of possible ways to use (7′) in an isobar analysis present themselves.

(i) Since A_j and B_j are supposed to have no physical singularities, we might take them as complex constants. This procedure doubles the number of parameters over the type of analysis in Ref. 2 and hence may not be practical.

(ii) Use (7′) at $q_j=0$ to give a set of equations for the B_j in terms of the A_j and τ_j, i.e.,

$$B_j = \sum_{k\neq j} (A_k + i B_k q_k^{l_k+1}) \tau_k^{(l_k)} q_k^{-l_k}\Big|_{q_j=0}.$$

(iii) Use a model such as our calculation[4,5] to determine B_j/A_j. Since the singularity in (7) comes from the last rescattering only,[6] a reasonable model can predict this ratio much more accurately than it can give A or B separately.

There are hopefully more and better ways to impose the constraint (7) yet to be discussed. Of the ways outlined above, (i) is the least biased, but the hardest to implement, and (ii) is probably the most attractive with (iii) being used as a test.

Before complicating an analysis by including the constraint (7), we should ask when is it numerically important. Since $\mathrm{Abs} f_i$ depends on $f_j\tau_j$ ($i\neq j$), it can be big if more than one isobar pair is important. This will be the case for identical particles since then $f_j\tau_j$ is just $f_i\tau_i$ of different arguments. It also occurs if two or more different pair groupings are important as in $\pi+N \to \pi+\Delta$ or $\rho+N$. However, if the isobars are narrow and the energy high, $f_j\tau_j$ evaluated at the q_j corresponding to q_i on or near the ith pair resonance may be very small, and the effect may be neglected. This is probably the situation in most high-energy missing-mass studies of meson resonances, but may not be in some moderate-energy studies. Unfortunately, the π-production analysis of Herndon *et al.*[2] is also not in such a regime and (7) is important to them. For example, the f_k amplitude for $\pi+N \to \Delta+\pi$, which they take to be constant, we would suggest should be written

$$f_k = A_k + iq_k^2 B_k ,$$

since Δ is a p wave. The q^2B term clearly varies rapidly (quadratically), but more importantly it is easily comparable to the A term over most of the phase space. This emerges both from a general consideration of the ranges involved and of the strengths of the competing ρN and σN channels, and is a feature of our dynamical calculation.[5] Similar difficulties occur in the study of threshold enhancements like the n-n scattering length in $n+d \to n+n+p$ since (7) is a threshold singularity. Here too we have done a numerical calculation that shows the rapid variation of f_i in the critical region.[6]

In conclusion we have shown that unitarity imposes a rapid variation on an amplitude usually assumed to be slowly varying in simple pairwise final-state-interaction fits to three-body final states. This causes difficulties both for threshold enhancements as in the neutron-neutron scattering-length problem and for isobar analysis of the type made by Herndon *et al.* for $\pi+N \to \pi+\pi+N$.[2] In numerical models we have also obtained this important variation. Neglect of this variation may invalidate the conclusions of Herndon *et al.* as to the relative sign of various π-N resonance decays to $\pi\Delta$ and ρN. Our numerical calculations[5] in fact give different signs from the ones they obtain in at least one case crucial to quark-model analysis.[1]

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[1]D. Faiman and J. Rosner, Phys. Lett. 45B, 357 (1973), and other references cited therein.

[2]D. J. Herndon *et al.*, Lawrence Berkeley Laboratory Report No. LBL-1065, and SLAC Report No. SLAC-PUB-1108, 1972 (unpublished).

[3]Cf. R. Aaron and R. D. Amado, Phys. Rev. 140, B1291 (1965).

[4]R. Aaron and R. D. Amado, Phys. Rev. Lett. 27, 1316 (1971), and references cited therein.

[5]R. Aaron and R. D. Amado, to be published.

[6]S. Adhikari and R. D. Amado, to be published.

[7]To prove (3) we use $2i\,\mathrm{Im}(AB) = AB - A^*B^* = AB - AB^* + AB^* - A^*B^* = 2iA\,\mathrm{Im}B + 2i(\mathrm{Im}A)B^*$.

[8]Cf. R. D. Amado, D. F. Freeman, and M. H. Rubin, Phys. Rev. D 4, 1032 (1971).

[9]R. D. Amado, Phys. Rev. 158, 1414 (1967).

[10]Also by R. Cahill, to be published.