Van der Pol Model for Unstable Waves on a Beam-Plasma System

Peter DeNeef*
Physics Department, University of California at San Diego, La Jolla, California 92037
and
Herbert Lashinsky†
Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742
(Received 17 August 1973)

The unstable modes on a low-density, cold-electron-beam–plasma system are treated as an ensemble of Van der Pol oscillators. This model predicts the observed amplitude limiting of unstable waves at the neighboring frequencies of a launched wave that traps the beam electrons.

The Van der Pol equation[1] describes an oscillation with an amplitude that exhibits spontaneous exponential growth followed by a stable saturation level. It has been used to describe mode locking[2] and frequency pulling[3] in Q machines, the multimode operation of lasers,[4] and suppression of the ion sound instability in an arc discharge.[5] Recently, it has been applied to the effect of beam modulation upon the standing waves on an electron-beam-produced plasma in which the end-plate potential reflects the electron beam back into itself.[6]

In this Letter we apply the Van der Pol model to the unstable waves that grow when a low-density, cold electron beam is injected into a collisionless plasma. We find qualitative agreement with recently published results on the amplitude limiting of unstable waves at the neighboring frequencies of a launched wave that traps the electron beam.[7]

When a low-density, cold electron beam is injected into a collisionless, Maxwellian plasma along a strong magnetic field, unstable waves in a narrow frequency band grow exponentially in space, starting from the thermally generated noise level. The growth continues at the rate predicted by linear theory until the fastest-growing-wave's amplitude is sufficient to trap the beam electrons. At that point the wave's growth saturates, followed by a slow spatial amplitude oscillation as the trapped beam electrons bounce back and forth in the potential wells of the wave.[8] Any wave within the unstable frequency band will exhibit this growth, saturation, and oscillation, provided that its initial amplitude it sufficiently large to allow that wave to trap the beam first. Consequently, during its exponential growth and initial saturation stage, each unstable eigenmode may be regarded as a Van der Pol oscillator.

The spatially evolving wave amplitude $\varphi_k(x)$ of each unstable mode is modeled by the Van der Pol equation:

$$\frac{d^2\varphi}{dx^2} - \alpha\frac{d\varphi}{dx} + \gamma\frac{d}{dx}(\varphi^3) + k_n^2\varphi = 0. \qquad (1)$$

For the fastest-growing mode,[8] for example, the spatial linear growth rate is

$$\alpha/2 = (3^{1/2}/2)k_n(\eta'/2)^{1/3} \ll k_n,$$

and the saturation amplitude $\varphi(x\to\infty)$ is

$$a_0 \equiv (4\alpha/3\gamma)^{1/2} \simeq mu^2(\eta')^{2/3}/e,$$

where $\eta' = \frac{1}{3}(n_b/n_0)(u/\bar{v})^2$, u is the beam velocity, $\bar{v}$ is the plasma electron thermal velocity, k_n is the wave number of the mode, $n_b/n_0 \ll 1$ is the ratio of the beam to plasma density, and e/m is the ratio of electron charge to mass.

We are interested in the case where an unstable wave of wave number k_1 is launched on a beam-plasma system with a sufficient amplitude to grow and trap the beam before any thermally generated, unstable noise modes can influence the beam dynamics significantly. If the launched wave is the fastest-growing mode, its behavior $\Phi(x)\sin(k_1x)$ is well known from the single-wave theory for a small, cold beam injected into a collisionless plasma.[8] The launched mode $\Phi(x)$ will act as a driving force that is detuned from resonance by the amount $k - k_1$. Thus, to determine the effect of $\Phi(x)$ on the unstable modes at nearby wave numbers, the behavior of these modes is modeled by the driven Van der Pol equation with $\Phi(x)\sin(k_1x)$ as the forcing function. To solve this equation for $\varphi_k(x_0)$, the spectrum at a chosen point in space, we treat $\Phi(x)$ as a constant $B = \Phi(x_0)$, using the fact that $(d\Phi/dx)/\Phi(x) \ll k_1$. This corresponds to replacing the spatially growing and saturating launched wave by a constant-amplitude driver wave:

$$\frac{d^2\varphi}{dx^2} - \alpha\frac{d\varphi}{dx} + \gamma\frac{d}{dx}(\varphi^3) + k_n^2\varphi = Bk_n^2\sin(k_1x). \quad (2)$$

Coupling among the equal-amplitude noise modes is ignored as a small effect compared to the driving force of the large, launched wave.

In the linear region [before the nonlinear term $\gamma d(\varphi^3)/dx$ becomes important], the solution of Eq. (2) is the sum of (a) the solution to the homogeneous equation, i.e., the exponentially growing mode at the natural oscillator wave number k_n, and (b) the particular solution, i.e., an oscillation at the driver wave number k_1. Correspondingly, in the linear region of the beam plasma instability (before the beam is trapped), a launched unstable wave has no effect on the exponentially growing noise modes.[7]

Once the nonlinear term in Eq. (2) grows to importance, the noise modes at k_n and the driven mode at k_1 no longer behave independently. Also, the eigenmodes of the system no longer have the simple form $\exp(ik_nx)$. Instead, as is evident by regarding each noise mode as a modulation of the launched-wave's amplitude and phase,[7] each eigenmode in the nonlinear region is a combination of the two terms $\exp[i(k_1+\delta k)x]$ and $\exp[i(k_1-\delta k)x]$. Because the predictions of the Van der Pol model are also symmetric about k_1, we continue to label each eigenmode by a single wave number k_n as in Eq. (2).

A standard method of treating Eq. (2) is to seek a solution of the form[9,10]

$$\varphi(x) = a(x)\sin(k_nx) + b(x)\sin(k_1x+\theta), \quad (3)$$

where d^2a/dx^2 and d^2b/dx^2 are negligible, $(da/dx)/a \ll k_n$, and $(db/dx)/b \ll k_1$. It is found that for driver amplitudes B sufficiently large, there exists a range of wave numbers centered on k_1 within which the oscillation at the natural wave number k_n is "asynchronously quenched,"[10] i.e., within which $a=0$. Elsewhere, the solution involves the combination of oscillations at k_1 and k_n. Outside the quenched wave-number band, Eqs. (2) and (3) lead to[9]

$$b^2\{[1-3(b/a_0)^2]^2 + [2(k_n-k_1)/\alpha]^2\} = (Bk_n/\alpha)^2 \quad (4)$$

and $a^2+b^2=a_0^2$, where a and b are the resulting amplitudes at the natural-oscillator wave number k_n and the driving wave number k_1 respectively, and $a_0=(4\alpha/3\gamma)^{1/2}$ is the "free running" amplitude at the natural wave number in the absence of a driving force, as defined above. For convenience in comparing the calculation with experiments in which $\varphi_k(x_0)$ is measured as a spectrum of frequencies, in Eq. (4) we make the transformation $\omega_n \simeq k_nu$, valid over the narrow spectrum of interest.

Curve A of Fig. 1 shows the result of a calculation of $a(\omega_n)$ using Eq. (4) with $a_0(\omega_n)$ specified as shown in curve B. $\alpha(\omega_n)$ was assumed constant ($=7.2\times10^7$ rad/sec) as was $(B\omega_n/a_0\alpha)^2=0.37$. The frequency range within which $a=0$ was taken to be $|\omega_1-\omega_n|/2\pi < 2$ MHz.

An experiment has been described previously[7] in which a 150-V, 200-μA electron beam is injected along a 180-G magnetic field into a plasma column with an effective electron density of 5×10^8 cm^{-3} ($\omega_p/\omega_c \simeq 0.4$) and an electron temperature of approximately 5 eV. When a wave at ω_1 is launched with sufficient amplitude to trap the beam immediately, the spectrum of beam-grown noise received downstream shows the same features as the calculated curve A. A frequency

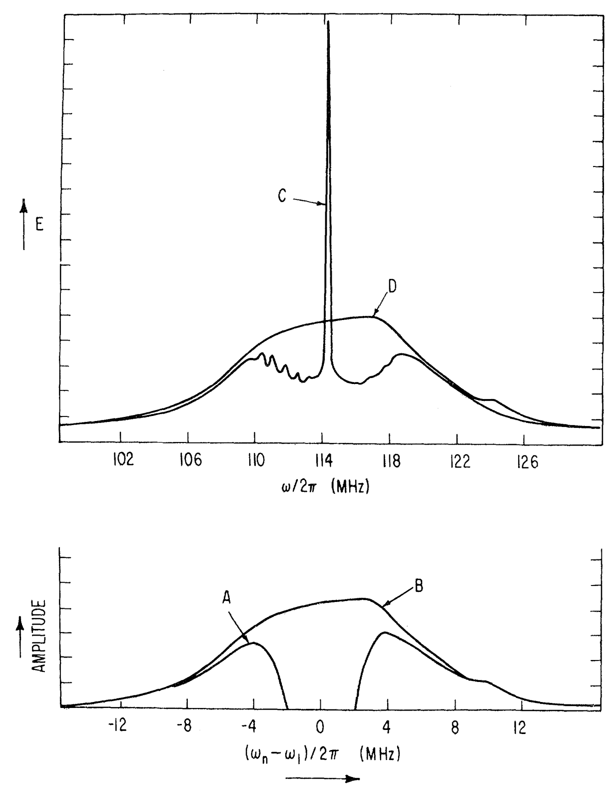

FIG. 1. Curve A, calculation using Eq. (4) with parameters given in the text and $a_0(\omega_n)$ as shown in curve B. Curve C, amplitude spectrum received at a position after launched wave has trapped the electron beam. Curve D, spectrum received at same position as C, no launched wave.

band centered on ω_1 is "quenched" or prevented from growing, and adjacent unstable noise modes are limited in amplitude. Of greater interest is the case when the launched wave is (a) small enough that its amplitude grows exponentially before saturating, and (b) far enough above the initial thermal noise elvel that the launched wave always traps the beam. The exponentially growing noise is observed to be unaffected until the launched wave has trapped the beam. Curve C is the spectrum received at a point after the beam has been trapped by a 114-MHz wave launched where the beam enters the plasma and with an amplitude approximately 10 dB above the thermal noise level. Curve D is the spectrum received at the same position but with no launched wave.

The similarity between the calculation and the experimental result is evident. The principal failure of the calculation is that the observed noise amplitude in the "quenched" region ($|\omega_1 - \omega_n| < 2$ MHz) is not zero, but rather is limited to the level it had grown to before trapping. This amplitude limiting and the spatial evolution of these limited waves have been previously calculated by regarding them as a slow modulation of the launched wave's amplitude and phase.[7] It is likely that the failure is not the fault of the Van der Pol model, but rather is a result of the constant-driver approximation that facilitated the calculation.

The principal success of the model is that, unlike the modulational point of view, the Van der Pol model clearly exhibits the frequency range over which a launched wave affects the unstable modes at neighboring frequencies.

In summary, the unstable modes on a low-density, cold-electron-beam–plasma system have been treated as an ensemble of Van der Pol oscillators. Qualitative agreement is found with the observed amplitude limiting of unstable waves at the neighboring frequencies of a launched wave that has trapped the electron beam.

*Present address: Department of Applied Sciences, University of California, Davis, Calif. 95616. Work at San Diego supported by National Science Foundation Grant No. GP-27120.

†Work supported by U. S. Atomic Energy Commission Contract No. AT(40-1)3405 and U. S. Air Force Office of Scientific Research Contract No. AFOSR 72 2179.

[1]B. van der Pol, Phil. Mag. 3, 65 (1927).

[2]H. Lashinsky, in *Nonlinear Effects in Plasmas*, edited by G. Kalman and M. Feix (Gordon and Breach, New York, 1969).

[3]H. Lashinsky, in *Turbulence of Fluids and Plasmas (Microwave Research Institute Symposia–Polytechnic Institute of Brooklyn, v. 18)*, edited by J. Fox (Wiley, New York, 1969).

[4]W. E. Lamb, Jr., Phys. Rev. 134, A1429 (1964).

[5]B. E. Keen and W. H. W. Fletcher, Phys. Rev. Lett. 24, 130 (1970).

[6]Y. Nakamura, J. Phys. Soc. Jap. 31, 273 (1971).

[7]C. P. DeNeef, J. H. Malmberg, and T. M. O'Neil, Phys. Rev. Lett. 30, 1032 (1973).

[8]T. M. O'Neil, J. H. Winfrey, and J. H. Malmberg, Phys. Fluids 15, 1514 (1972).

[9]J. J. Stoker, *Nonlinear Vibrations* (Interscience, New York, 1950), pp. 149–187.

[10]E. M. Dewan, IEEE Trans. Automat. Contr. 17, 655 (1972).

Electrostatic Wave Reflection from a Plasma Density Gradient*

C. W. Roberson, A. S. Ratner, and J. L. Hirshfield
Department of Engineering and Applied Science, Yale University, New Haven, Connecticut 06520
(Received 11 June 1973)

We report measurements of the reflection coefficient R for electron plasma waves propagating into a density gradient on a narrow column. Within experimental error, R is proportional to $\exp(-\delta)$ when R is less than 0.25 and is equal to $\exp(-0.23\delta)$ for R greater than 0.25, where δ is the inverse of the WKB parameter.

Considerable theoretical effort has recently been applied to the problem of electrostatic plasma-wave reflection from plasma density gradients. One motivation for this effort has been to determine if reflections of convectively unstable waves from the ends of a mirror machine affect the stability of these devices.[1-6] The problem is also relevant to considerations of the axial boundary conditions for beam-plasma devices. In this Letter we present measurements of the reflec-