[12]It might be thought that an effective three-phonon process can be built out of successive occurence of a four-phonon and a five-phonon vertex, say. However, it can be shown that the temperature dependence of such a process tends to zero more strongly than the T^4 dependence characteristic of the three-phonon process.

[13]The form of the spectral function is easily seen to affect the temperature-frequency dependence of the acoustic attenuation. For, let us calculate the lowest-order perturbation-theoretical correction to the Green's function $D^R(\boldsymbol{k},\omega)=D^{(0)}+D^{(1)}$, and use it to find the spectral distribution $\rho(\boldsymbol{k},\omega)=-\operatorname{Im} D^R(\boldsymbol{k},\omega)/\pi$, for real ω. If this calculated $\rho(\boldsymbol{k},\omega)$ is substituted in the Pethick–ter Haar formula, then, for a convex spectrum the resulting attenuation is that of the four-phonon process, i.e., $\alpha \propto \omega T^6$.

[14]J. Jäckle and K. W. Kehr [Phys. Rev. Lett. 27, 654 (1971)] suggest that the spectrum becomes convex at progressively smaller values of $\boldsymbol{k}$ with rising pressure, which explains the shoulder in the high-pressure acoustic attenuation. There is no conflict with the high-pressure data of Phillips, Waterfield, and Hoffer (Ref. 5) since at high pressure the region of concavity is too small to contribute appreciably to the specific heat.

Thermal Anomalies of He^3: Pairing in a Magnetic Field*

Vinay Ambegaokar and N. D. Mermin
Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York 14850
(Received 27 November 1972)

It is shown that the linear splitting in a magnetic field of the thermal anomaly (*A*) observed recently in liquid He^3 at about 2.5 mK can be explained by postulating pairing in a state of odd relative angular momentum. Further structure is predicted, and implications of the theory and experiment are discussed.

We have been studying the general problem of BCS pairing of neutral spin-$\frac{1}{2}$ particles in a magnetic field H, with a view of explaining the recently reported thermal anomalies in liquid He^3.[1,2] Though far from an understanding of the full H-T phase diagram, we have found structure in the neighborhood of the normal-superfluid phase boundary which has been observed experimentally.[3] The reason for this preliminary report is that our analysis predicts additional phase boundaries, observation of which would aid further theoretical diagnosis of the quite complex possibilities farther from the normal-superfluid phase boundary, and also because the structure already observed provides evidence that the ordered state is more likely pairing of the BCS type than itinerant antiferromagnetism.

We use conventional weak-coupling of BCS theory with a pairing interaction of the form

$$\langle \vec{k}, -\vec{k}|V|\vec{k}', -\vec{k}'\rangle = v(\hat{k}\circ\hat{k}') = \sum v_L P_L(\hat{k}\circ\hat{k}'), \quad |\epsilon(\vec{k})-\epsilon_F|, |\epsilon(\vec{k}')-\epsilon_F| < \omega \lesssim \epsilon_F; \\ = 0, \text{ otherwise.} \tag{1}$$

In contrast to the assumption usually made in treatments of metallic superconductors, the energy variation of the level density $N(\epsilon)$ turns out to be important near T_c in a magnetic field. Consequently, the naive energy dependence assumed in (1) must eventually be reexamined. We suspect that more elaborate forms will at worst affect only the value of the numerical constants in Eq. (3) below.

At any field H, the temperature $T_c^{(1)}(H)$ at which ordering first appears is determined by the linearized gap equation, whose solutions can be classified by L. Even-L pairing gives a transition temperature $T_c^{e}(H) = T_c - O(\gamma^2 H^2/k_B^2 T_c)$ which falls quadratically and quite strongly with H, in contrast to what is observed.[4]

The observed phase boundary is almost an isotherm, but at high fields two transitions are seen separated in temperature by a small term linear in H.[5] A simple picture is suggested: odd-L pairing occurring independently within each spin population, first $[T_c^{(1)}(H)]$ among favorably aligned spins, then $[T_c^{(2)}(H)]$ among unfavorably aligned ones. A rough estimate based on allowing the level density to shift with field gives

$$T_c^{(1,2)}(H) \approx \omega \exp\{-[vN(\epsilon_F \pm \gamma H)]^{-1}\} \approx T_c[1 \pm \gamma H (N'/N)\ln(\omega/T)], \tag{2}$$

which has about the observed order of magnitude if N is estimated by its free-particle form.

Our quantitative analysis of odd-L pairing bears out this picture to a considerable degree, with some important reservations. The highest transition temperature at fixed field is indeed raised from its zero-field value by a small term linear in H, present [assuming Eq. (1)] solely because of the energy dependence of the level density, and signifying a transition into a phase (A_1) in which only a single spin population is paired.

Below $T_c^{(1)}(H)$ (at fixed field) one must use the nonlinear gap equation, but because the observed splitting of the A transition is small, one may safely retain only the next nonvanishing order. It is almost certain that this close to the phase-boundary, mixing between different (odd and/or even) L can still be ignored, unless the v_L's for two distinct L's happen to be nearly degenerate.[6] Assuming that the next transition is second order, we find it occurs at a temperature $T_c^{(2)}(H) = 2T_c - T_c^{(1)}(H)$, below which ($A_2$ phase) the liquid contains two independently paried spin populations [see (iii) below].

Below $T_c^{(2)}(H)$ an exhaustive test of the A_2 phase for second-order instabilities is difficult for $L \geqslant 3$, even when only cubic terms are retained in the gap equation. However, in the p-wave case we can rigorously locate the next (as yet unobserved) instability at a temperature $T_c^{(3)}(H)$ [given in (iv) below] which heralds the onset of a third phase (A_3) in which both spin populations are coherently mixed.[7]

This discription of three phases is subject to the following *caveat:* One cannot, in general, exclude the possibility that the fluid may go from A_2 to A_3 or directly from A_1 to A_3 via a first-order transition. This happens [see (iv) below] in the p-wave case, where the second order A_2-A_3 phase boundary becomes unstable at a tricritical point, at subkilogauss fields but within microdegrees of T_c. The associated first-order boundary will be difficult to resolve from the normal-A_1 phase boundary, but the tricritical point may well be observable as a divergent specific-heat discontinuity [see (v) below] across the (unresolved) A_1-A_2-A_3 phase boundaries as H drops to its tricritical value. Similar complications seem likely should the ordered phase have higher L.

The following points deserve emphasis:

(1) The observed linear splitting of the A transition favors the pairing model over the spin-density-wave hypothesis. In a conventional spin-density wave both spin populations are mixed, as in the A_3 phase. One could also contemplate itinerant antiferromagnetism with independent density oscillations in each spin population 180° out of phase (analogous to an A_2 phase, but with order parameters of equal magnitude) but for a single spin population to be paired without a compensating pairing in the other (as in the A_1 phase) would require an accompanying mass-density wave, which would appear to be prohibitively expensive in energy.

(2) Observation of the A_3 transition would greatly increase one's confidence in this model of the equilibrium state. Measurements are required in the range from about 3 kG (below which the transition is too close to A_1 and A_2 to be distinguishable) to 20 kG (above which the transition is too low in temperature for current observation). The associated specific-heat discontinuity should be about $\frac{1}{5}$ that associated with each of the A_1 and A_2 transitions in this field range.

(3) Observation of or failure to observe the tricritical point would be informative.

(4) We are unable to resolve a difficulty in the pairing model pointed out by Leggett.[8] Our analysis predicts a weakly temperature-dependent static susceptibility in the A_1 and A_2 phases, but an appreciable temperature dependence[9] in the A_3 phase. The resonance experiments,[2] however, indicate little temperature dependence though according to our model they are done in the A_3 phase. Concerns of this sort led Leggett to suggest that the ordered phase must have a very high L. According to our best estimates, however, $L = 3$ pairing yields a temperature dependence in the A_3 phase quite comparable to that for $L = 1$, and we do not believe this should change with still higher L. If we assume the resonance experiments are correctly interpreted, there are other possible resolutions of the problem. Conceivably, the pressure dependence of the relevant v_L may yield a T_c that declines with increasing pressure along the melting curve. This would mean that $T_c - T$ was not as large as the absolute temperature drop. Constant-pressure experiments could settle this point. Alternatively, at temperatures so low that the theoretical susceptibility for the A_3 phase has declined appreciably, there is little justification for ignoring the coupling to other L. Extrapolation of our results near the phase boundary to lower T is then quite unwarranted, and the correct susceptibility must be sought in a far more complex analysis.[10]

We conclude with a summary of the analytic basis for these results: For given odd L we derive a free-energy functional which near the transition temperature, and for fields $H \ll \pi k_B T_c/\gamma \sim 10^5$ G, takes the form

$$F = -N(0)[\tfrac{1}{2}(t+\eta h)\langle|\Delta_\uparrow|^2\rangle + \tfrac{1}{2}(t-\eta h)\langle|\Delta_\downarrow|^2\rangle + (t-2\beta h^2)\langle|\Delta_0|^2\rangle - \tfrac{1}{2}\beta\langle\tfrac{1}{2}|\Delta_\uparrow|^4 + \tfrac{1}{2}|\Delta_\downarrow|^4 + |\Delta_0|^4 + 2|\Delta_0|^2|\Delta_\uparrow|^2 + 2|\Delta_0|^2|\Delta_\downarrow|^2 + 2\,\mathrm{Re}(\Delta_0^2\Delta_\uparrow\Delta_\downarrow)\rangle]. \quad (3)$$

Here $\Delta_\uparrow$, $\Delta_\downarrow$, and $\Delta_0{}^*$ are the $\uparrow\uparrow$, $\downarrow\downarrow$, and $\uparrow\downarrow$ matrix elements of the (symmetric) order parameter and are linear combination of spherical harmonics of degree L; the bracket denotes an angular average; t is $(T_c - T)/T_c$; β is $7\zeta(3)/8\pi^2 = 0.11$; $h = \gamma H/k_B T_c$; $N(0)$ is the density of levels for one spin population at the Fermi surface; and

$$\eta = k_B T_c N'(0)\frac{\ln(1.14\omega/T_c)}{N(0)},$$

in agreement with the estimate (2).[11] Because $N'(0)$ is unknown, we regard η as a parameter to be fitted to the observed splitting.

Minimizing the free energy (3) gives

(i) For $t < -\eta h$: $\Delta_0, \Delta_\uparrow, \Delta_\downarrow = 0$.

(ii) For $t < 2\beta h^2$, $-\eta h < t < \eta h$: $\Delta_0, \Delta_\downarrow = 0$, $\langle|\Delta_\uparrow|^2\rangle = (t+\eta h)/\beta A$, $F = -N(0)(t+\eta h)^2/4\beta A$, where A denotes the nimimum value of $\langle|Y|^4\rangle/\langle|Y|^2\rangle^2$, over all angular functions Y in the appropriate L manifold. (The best Y gives the angular dependence of $\Delta_\uparrow$.) For p waves one can prove that the best Y is Y_{11}, giving $A = \frac{6}{5}$. For f waves we have not found a choice better than $Y = Y_{32}$ ($A = 210/143$).

(iii) For $t < 2\beta h^2$, $t > \eta h$: $\Delta_0 = 0$, $\langle|\Delta_\uparrow|^2\rangle = (t+\eta h)/\beta A$, $\langle|\Delta_\downarrow|^2\rangle = (t-\eta h)/\beta A$, $F = -N(0)(t^2 + \eta^2 h^2)/2\beta A$, and $\Delta_\uparrow$ and $\Delta_\downarrow$ have the same angular dependence (Y) except for their relative orientation, which is arbitrary.

(iv) For $t > 2\beta h^2$ we find, for $L = 1$, that a second-order transition to $\Delta_0 \neq 0$ first becomes possible along the line (Fig. 1) $(4\beta h^2)^2 + (\eta h)^2 = t^2$, with $\Delta_\uparrow \propto -u + iv$, $\Delta_\downarrow \propto u + iv$, $\Delta_0 \propto w$, where u, v, and w are three orthogonal projections of $\hat{k}$. This line, however, is only stable against small fluctuations when $4\beta h^2 > t(\sqrt{10} - 2)/3$. When the inequality is reversed, we find by numerical computation a first-order line which joins smoothly onto the second-order line; i.e., there is a tricritical point given by $4\beta h^2/t = 0.387$, $\eta h/t = 0.922$. The first-order line crosses the second-order line $t = -\eta h$ at $H = 0.48 H_t$. If η is adjusted to fit the observed splitting [taken as (0.005 mK)H(kG)], we find $H_t \sim 1$ kG, $T_c - T_t \sim 2.5$ μK. We have not rigourously excluded further first-order lines in the region $t > 2\beta h^2$, but a somewhat cursory numerical investigation has not revealed any.

(v) The specific-heat discontinuities predicted for the second-order transitions in the p-wave case are

$$\Delta C_1 = \Delta C_2 = 5\Delta C/12,$$

$$\Delta C_3 = \frac{5\Delta C}{6}\left[3\left(\frac{4\beta h^2}{t}\right)^2 + 4\frac{4\beta h^2}{t} - 2\right]^{-1},$$

where ΔC is the zero-field discontinuity. Note the singularity at the tricritical point.

(vi) For f-wave pairing the structure at kilogauss fields (away from the region of first-order phase boundaries) appears similar to that described above, but we do not yet have a rigorous solution to the angular minimization problem.

We have greatly benefitted from frequent discussions with D. M. Lee, R. C. Richardson, and their associates. We are also grateful for advice and help from V. J. Emery, M. E. Fisher, R. B. Griffiths, A. J. Leggett, S. G. Cochran, J. W. Serene, G. B. Stare, and P. Wölfle.

*Work supported in part by the U. S. Office of Naval

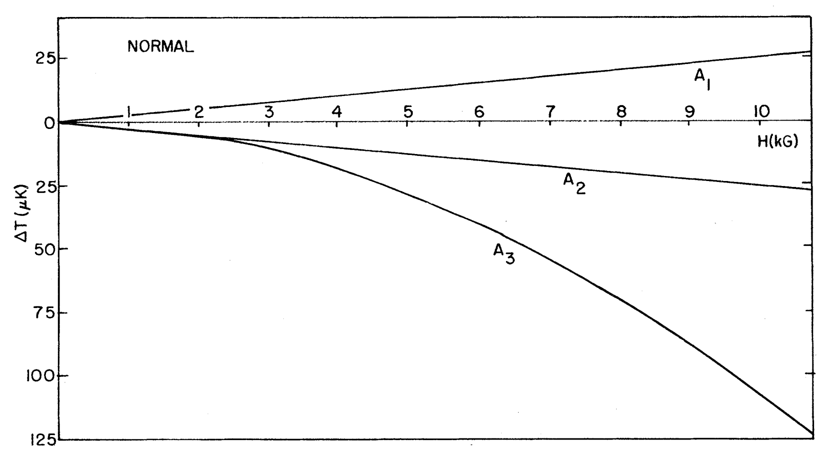

FIG. 1. Phase diagram for p-wave pairing. The parameter η has been adjusted to give a splitting of 0.005 mK at 1 kG between the A_1 and A_2 transitions. The A_3 transition is second order for fields greater than 1 kG. At lower fields the phase boundaries have the complicated behavior described in (iv), but these details cannot be resolved on the scale of the figure.

Research under Contract No. N00014-67-A-0077-0010, Technical Report No. 30, and by the National Science Foundation through Grant No. GP 27355. Additional support was received from the National Science Foundation (Grant No. GH 33637) through the Materials Science Center, Cornell University, Report No. 1918.

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[2]D. D. Osheroff, W. J. Gully, R. C. Richardson, and D. M. Lee, Phys. Rev. Lett. 29, 920 (1972).

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[4]Even-L pairing is independently excluded by the existence of a NMR shift in the A phase (Ref. 2) which singlet pairing cannot account for.

[5]The measurements in Refs. 3 reveal only the relative splitting of the A transition.

[6]One should not overlook the possibility of degeneracy, given the likelihood that the v_L vary considerably with pressure.

[7]We find a similar instability when $L=3$, but have not demonstrated that it is not preceded by a different second-order transition.

[8]A. J. Leggett, Phys. Rev. Lett. 29, 1227 (1972).

[9]The temperature dependence well below the A_2-A_3 phase boundary is essentially that described by R. Balian and N. R. Werthamer [Phys. Rev. 131, 1553 (1963)].

[10]For similar reasons we are unprepared to offer a specific model for the B transition, since this requires testing the stability of the best A_3 phase well below the normal-superfluid transition temperature. Some specific possibilities have been mentioned by P. W. Anderson and C. M. Varma (to be published).

[11]This formula and Eq. (3) have been somewhat simplified by using $\omega \gg T_c$.

Nonlinear Mode Competition in Beam-Plasma Instability*

W. Carr, D. Bollinger, D. Boyd, H. Liu, and M. Seidl
Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030
(Received 28 September 1972)

In a cold beam-plasma system unstable for $f > f_{ce}$, two waves are launched, one at f_0, with large growth rate, and a small test wave at f_T, with small growth rate. The wave at f_0 saturates because of beam trapping, independent of the test-wave amplitude. After the beam is trapped, the test wave ceases growing and exhibits amplitude oscillations in phase with large wave. This is consistent with a linear interaction of the test wave with the modified electron beam.

Recent work on cold beam-plasma instabilities shows the trapping of beam particles in the wave potentials to be the dominant nonlinear saturation process. Gentle and Roberson[1] observed a narrow wave spectrum at the onset of saturation which showed the amplitude oscillations characteristic of beam trapping in a single wave. More recently, Mizuna and Tanaka,[2] Bollinger *et al.*,[3] and Gentle and Roberson[4] observed changes in the beam distribution function which are in good qualitative agreement with theoretical work, particularly recent one-dimensional computer calculations for beam trapping by a single wave.[5-9]

Of interest here is the subsequent nonlinear development, which is not as clearly established. Onishchenko *et al.*[5] and O'Neil, Winfrey, and Malmberg[6] discuss the later nonlinear picture and state that eventually a broad wave spectrum will develop governed by quasilinear theory. There are several possible mechanisms to generate the broad wave spectrum, the most straightforward one being the continued growth of those frequency components which are small at the point of saturation of the principal wave.

In the present experiment we examined the behavior of a small-amplitude test wave (frequency f_T) in the presence of a large-amplitude wave (frequency f_0) in a cold beam-plasma system, $f_0, f_T > f_{ce}$. Briefly, we find that the test wave does not continue to grow after saturation, but remains small. Thus, the small components of the spectrum do not lead to a transition to quasilinear behavior in a straightforward way.

The experiments were done in a machine described elsewhere[10]; the system parameters are as follows: axial background field $B_0 = 180$ G ($f_{ce} = 0.5$ GHz), plasma frequency $f_p = 0.6$ GHz, plasma temperature ~3 eV, beam energy 350 eV, beam density $n_B \cong 2\times10^{-3} n_p$, beam diameter 6 mm, plasma diameter 5 cm, and interaction region ~1 m (50 cm used in the experiment has $\Delta n_p / n_p < 5\%$). The waves are launched from a transmitter near the electron gun and detected by a movable axial probe and tuned receiver. In this experiment the beam density is kept small enough so that amplified noise does not grow to