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Two-Stream Instability Heating of Plasmas by Relativistic Electron Beams*

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We discuss the two-stream interaction of relativistic electron beams with a plasma in the nonlinear regime supported by computer simulation experiments.

For relativistic electron beams with current I in excess of the critical Alfvén current I_A, i.e., $I/I_A = \nu/\gamma \gg 1$, the dominant mechanism by which the beam transfers its energy to the plasma is expected to be that due to the turbulent decay of the return current proposed by Lovelace and Sudan.[1] Here we use conventional notation: $\nu = Nr_e$, where N is the number of beam electrons per unit length and r_e is the classical electron radius; γ is the beam energy in rest mass units. On the other hand, for beams with $\nu/\gamma \sim 1$ the competing process of the electrostatic two-stream interaction becomes important. The present Letter is devoted to a discussion of the two-stream interaction, supported by computer simulation studies for a homogeneous beam-plasma system.

The gross magnetohydrodynamic stability of the beam can be assured on the time scale of the two-stream instability provided the beam propagates along a guide magnetic field such that the beam kinetic-energy density is less than the magnetic energy density.[2] In addition, the beam velocity distribution is assumed to satisfy $\Delta v_\parallel/c \ll \gamma_0^{-1}(n_b/2n_e)^{1/3}$, where n_b and n_e are the respective beam and plasma particle densities, $\Delta v_\parallel$ is the velocity spread in the direction of the average beam velocity $\vec{v}_0$, and $\gamma_0 = (1 - v_0^2/c^2)^{-1/2}$. Furthermore, we shall assume a one-dimensional spectrum for the unstable waves. There are four situations where the last assumption is valid: (i) beam propagation along a sufficiently strong magnetic field; (ii) $\Delta v_\perp/c \gtrsim (n_b/2\gamma_0 n_e)^{1/3}$, which reduces the growth rates of modes with finite[3,4] $k_\perp$ below $\gamma_0^{-1}(n_b/2n_e)^{1/3}$, where $\Delta v_\perp$ is the velocity spread perpendicular to $\vec{v}_0$; (iii) the plasma density profile has a minimum at the beam axis, and waves with finite $k_\perp$ will be reflected towards the direction of beam propagation; (iv) the beam has been modulated at a wavelength for maximum growth of the two-stream instability.

Under the above restrictions, the dispersion relation for electrostatic waves is

$$1 - \omega_e^2/\omega^2 - \omega_b^2/(\omega - kv_0)^2 = 0, \qquad (1)$$

where $\omega_e^2 = 4\pi n_e e^2/m$ and $\omega_b^2 = 4\pi n_b e^2/\gamma_0^3 m$. After N e-folds in amplitude, the width of the unstable spectrum predicted by (1) is $\delta k/k_0 \sim \gamma_0^{-1}(n_b/2n_e)^{1/3} \times N^{-1/2}$, where $k_0 \approx \omega_e/v_0$ is the wave number corresponding to the most unstable wave. It is clear that the spectrum can be very narrow when a high-energy beam interacts with a cold plasma. In this case, the spectrum may be approximated by a single wave with wave number k_0, the so-called single-wave model.

There are two main questions to be resolved: (i) the maximum intensity to which the unstable waves grow in the two-stream instability phase of the interaction, and (ii) the rate of energy transfer to the plasma after wave saturation. To characterize the strength of the interaction, we define the parameter $S \equiv \beta_0^2\gamma_0(n_b/2n_e)^{1/3}$. Several investigators have examined the regime $S \ll 1$. From a quasilinear analysis of a one-dimensional spectrum Fainberg, Shapiro, and Shevchenko[3] obtain for the saturation amplitude

$$W \equiv \sum_k |E_k|^2/8\pi n_b\gamma_0 mc^2 = 0.158S,$$

while Kovtun and Rukhadze[5] obtain $W = 0.198S$ using the single-wave model for the interaction.

We present a qualitative picutre of the single-wave model under the restriction $n_b/n_e \ll 1$ with S ranging from 0 to, say, 5. (Our results may indeed be valid for higher S, but further investigation is needed.) It is well known that the wave saturates by trapping the beam electrons. During the last couple of e-folds in wave amplitude, the "instantaneous" trapping frequency of those beam electrons, whose wave-frame kinetic energy is small compared to wave potential energy, rises rapidly. The energy exchange between these electrons and the wave maximizes approximately when they have completed about $\frac{1}{2}$ revolution in phase space. If $f_s(x,p)$ is the beam momentum distribution in the *wave frame* at the time of wave saturation, then the energy loss of the beam electrons averaged over a wavelength λ, as observed in the lab frame, is given by

$$\Delta\epsilon = n_b mc^2\gamma_w \int_{-\lambda/2}^{\lambda/2}(dx/\lambda)\int_{-\infty}^{\infty} dp\, f_s(x,p) \times\{\gamma_r - \gamma(x,p) + \beta_w[p_r - p(x,p)]\}, \quad (2)$$

where mcp_r and γ_r are the initial beam momentum and energy in the wave frame, $\beta_w = \omega_0/k_0c$, and $\gamma_w = (1-\beta_w^2)^{-1/2}$. Half of this energy loss can be assumed to furnish the mean electric field energy of the wave. The remainder is absorbed by the oscillatory motion of the nonresonant plasma electrons. Thus the mean energy density in the wave at saturation is

$$W \equiv |E_0|^2/16\pi n_b mc^2\gamma_0 = \tfrac{1}{2}\Delta\epsilon/n_b mc^2\gamma_0. \quad (3)$$

Let $\Delta\epsilon_1$ be the contribution from the $\gamma_r - \gamma(x,p)$ term and $\Delta\epsilon_2$ from $\beta_w[p_r - p(x,p)]$; $\Delta\epsilon_1$ represents the spread in energy, and $\Delta\epsilon_2$ is the change in the mean drift energy. As a result of wave growth the magnitudes of $\Delta\epsilon_1$ and $\Delta\epsilon_2$ both increase from their initial values; however, $\Delta\epsilon_1 < 0$, while $\Delta\epsilon_2 > 0$. For $S \ll 1$, the energy spread $|\Delta\epsilon_1| \ll |\Delta\epsilon_2|$, and one can regard the bulk of the beam electrons to rotate rigidly[6] in x-p space so that the beam which was initially described by $f(x,p) = \delta(p - p_r)$ is now, half a revolution later, represented by $f_s(x,p) \approx \delta(p + p_r)$. With this distribution we observe that $\Delta\epsilon_1 \approx 0$, and $\Delta\epsilon_2 \approx S/(1+S)$, which gives

$$W = \tfrac{1}{2}S(1+S)^{-3/2} \approx \tfrac{1}{2}S \text{ for } S \ll 1, \quad (4)$$

noting that $\gamma_w/\gamma_0 \approx (1+S)^{-1/2}$. For $S \gtrsim 1$ the energy spread $|\Delta\epsilon_1|$ becomes important, and the rigidly rotating-beam model is not valid. If indeed we calculate $\Delta\epsilon_1$ and $\Delta\epsilon_2$ by assuming that the wave amplitude reaches its final value suddenly, the energy loss $\Delta\epsilon$ scales more like $S(1+S)^{-2}$ compared to $S(1+S)^{-1}$ for the rigid-rotor model, and we obtain

$$W = \tfrac{1}{2}S(1+S)^{-5/2}. \quad (5)$$

Initially a beam electron has energy $mc^2\gamma_0$, where $\gamma_0 = \gamma_w\gamma_r(1+\beta_r\beta_w)$ (expressed in terms of wave-frame quantities). After half a trapping oscillation in phase space its energy is reduced to $mc^2\gamma_f$ with $\gamma_f = \gamma_w\gamma_r(1-\beta_r\beta_w)$. The energy loss is $mc^2(\gamma_0 - \gamma_f) = 2mc^2\gamma_w\gamma_r\beta_r\beta_w \approx mc^2\gamma_0 S(1+S)^{-1}$. Now all the beam electrons in fact do not lose this energy since some gain energy through being accelerated by the wave. If we denote by n_{eff} the effective number of electrons losing $mc^2(\gamma_0 - \gamma_f)$, i.e., $\Delta\epsilon = n_{\text{eff}} mc^2(\gamma_0 - \gamma_f)$, then

$$n_{\text{eff}}/n_b = (\gamma_w/\gamma_0)^3. \quad (6)$$

Thus because of the increased energy spread caused by strong relativistic beams, only $n_b(\gamma_w/\gamma_0)^3$ electrons can be considered to coherently rotate in the phase space and pump the wave.

As noted above, some beam electrons gain energy through being accelerated by the wave. In the high-energy regime an estimate of the maximum energy an electron can have is obtained as follows. The space-averaged momentum distribution may be assumed to be constant from $p \simeq 0$ to some $p = p_{\max}$ at the time of wave saturation. In view of (5) the energy loss to the waves can be

TABLE I. Summary of a series of one-dimensional computer simulations. Columns 2 and 3 give the basic parameters for each run. In column 4 the strength parameter $S \equiv \beta_0^2\gamma_0(n_b/2n_e)^{1/3}$ corresponding to each run is given. $W \equiv |E_0|^2/16\pi n_b\gamma_0 mc^2$ is given in column 5. The fraction of initial-beam kinetic energy gained by the plasma electrons after the second stage is denoted by K. R_5 and R_6 were not run long enough to complete the second stage. The initial plasma electron temperature is T_e. In all the runs, except R_5, which had a mass of 500, the ratio was 2000. There were a total of 40 000 particles: 20 000 ions, 18 000 electrons, and 2000 beam electrons.

Run	γ_0	n_b/n_e	S	W	K	T_e
R_3	2	0.050	0.435	0.0685	0.25	1.5 keV
R_4	2	0.110	0.555	0.0615	0.26	0.1 eV
R_5	4	0.001	0.296	0.0460	···	0.1 eV
R_6	4	0.001	0.296	0.0485	···	0.1 eV
R_7	4	0.010	0.617	0.0910	0.26	0.1 eV
R_8	4	0.010	0.617	0.0845	0.29	1.5 keV
R_9	4	0.050	1.150	0.0650	0.26	1.5 keV
R_{10}	8	0.010	1.310	0.0720	0.31	1.5 keV
R_{11}	8	0.050	2.320	0.0515	0.27	1.5 keV

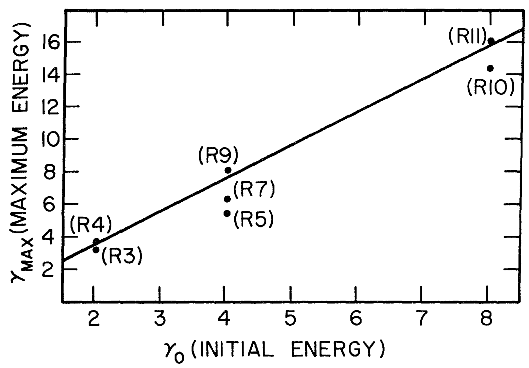

FIG. 1. Maximum energy obtained by an electron through being accelerated by the wave plotted as a function of initial energy γ_0. Solid line, Eq. (7); R_n refers to simulation runs given in Table I.

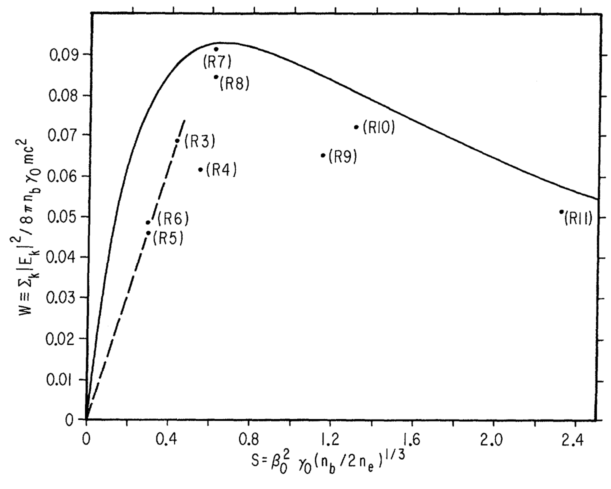

FIG. 2. Electric field energy in the wave spectrum at saturation given as a function of S. Dashed line, quasilinear result; solid line, Eq. (5); R_n refers to simulation runs given in Table I.

neglected to a first approximation, and from conservation of energy we find that

$$\gamma_{max} \approx 2\gamma_0 - \ln(4\gamma_0)/2\gamma_0. \qquad (7)$$

A short summary of a series of one-dimensional computer simulations is presented in Table I. We observe three stages in the evolution of the beam-plasma system. In the first stage the wave amplitudes grow exponentially at a rate given by linear theory, and after a short interval of time the amplitude of the wave with the fastest growth rate is such that it dominates the plasma dynamics. Thus T_e, T_i increase adiabatically at the same rate as the wave energy. This state comes to an end when the wave amplitude saturates abruptly. The maximum energy of electrons accelerated by the wave is plotted as a function of initial beam energy in Fig. 1. The results are in good agreement with (7). The electric field energy W at the time of wave saturation[7] is plotted as a function of S is Fig. 2. For $S<0.6$ the numerical results are in agreement with the quasilinear theory of Fainberg, Shapiro, and Shevchenko.[3] For $S<0.6$ the single-wave model is not strictly valid since 75% of the electric field energy is in the dominant wave compared to 90% for $S>0.6$. Although Eq. (5) does not fit the data closely for $S<0.6$, it describes the overall data quite well, including the maximum near $S=0.66$.

In the second stage, after saturation, the wave amplitude is seen to fluctuate at the trapping frequency of the particles at the bottom of the potential well. After at most two wave energy oscillation periods, the wave spectrum begins to decay (see Fig. 3). At this point, the large-amplitude wave acts as a pump wave to drive the oscillating two-stream instability. Note the sudden rise of ion fluctuations in Fig. 3. The observed growth rates of the unstable waves are in fair agreement with the theoretically predicted growth rates except at high wave numbers when Landau damping becomes strong. These unstable waves have low phase velocities since their frequency is $\sim\omega_e$, and their wavelength is determined by the ion fluctuations to be about $k\lambda_{de}\sim 0.2$. There is an efficient transfer of energy through this instability to the electrons in the tail of the distribution. Thus during this stage, the energy in the large wave eventually ends up in the plasma by the physical process described above, and the total amount of energy transferred to the plasma until the end of the second stage should be twice the maximum wave energy density $2n_b mc^2\gamma_0 W$. In the simulation experiments the energy transferred to the plasma (see Table I, column 6) appears to be in excess of this value. This can be explained in the following terms. The hydrodynamic phase of the two-stream instability ends when the large amplitude wave saturates, and the beam simultaneously acquires a large momentum spread. At this point the two-stream instability enters the kinetic phase. New waves grow in that region of momentum space where df/dp for the beam is positive, i.e., at the lower end of the beam momentum distribution. These waves with phase velocities much below the primary spectrum also extract energy from the beam which subsequently ends up in the tail of the plasma electron distri-

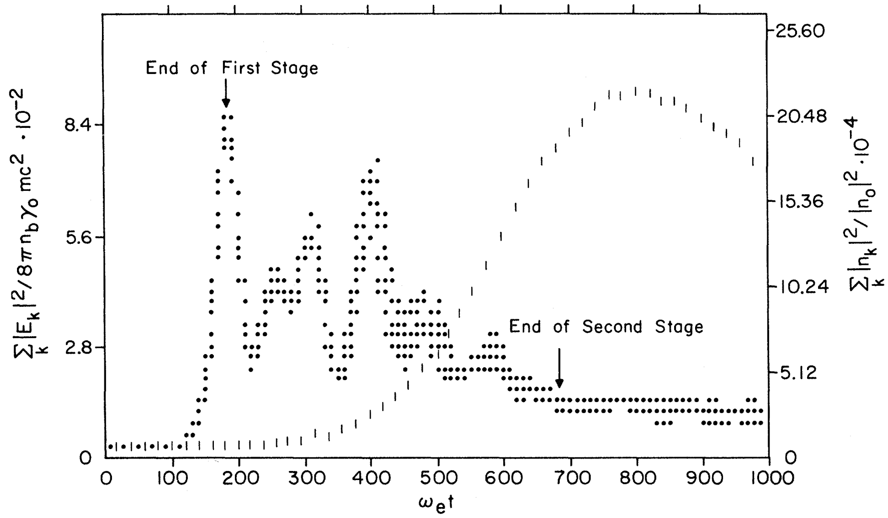

FIG. 3. Total wave electrostatic energy W (denoted by dots) and ion density fluctuations $\sum_k |n_k|^2/|n|^2$ given as a function of time in units of ω_e^{-1} for R_8. Different stages in the interaction are indicated. The rise in ion density fluctuations is coincident with the beginning of the wave spectrum decay.

bution.

In fact from Table I, column 6, we observe that roughly 25 ~ 30% of the initial beam kinetic energy is transferred to the plasma electrons in times of the order of a few hundred ω_e^{-1}.

In conclusion, we wish to reiterate that our calculations are based on a relatively simple model and do not include such effects as density gradients along the beam axis.

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